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Algebraic Geometry Summer Term 2018

Exercise Sheet 10. Hand in by Friday, June 29.

Exercise 1 (Noether Normalization, Refined Version). Let S be a finitely generated K-algebra, and let $I \subset S$ be an ideal. There exist integers $\delta \le d$ and a Noether normalization $K[y_1, \dots, y_d] \subset S$ such that

$$I \cap K[y_1, \dots, y_d] = (y_1, \dots, y_\delta),$$

in other words we can map $V(I)$ onto a linear variety.

Exercise 2. Let I be a proper ideal of $K[x_1, \dots, x_n]$, and let $>$ be a global monomial order on $K[x_1, \dots, x_n]$. Suppose that, for some c, the following two conditions hold:

(1) $in(I)$ is generated by monomials in $K[x_1, \dots, x_c]$
(2) $in(I) \supset (x_1, \dots, x_c)^m$ for some m.

Prove that the composition

$$R = K[x_{c+1}, \dots, x_n] \hookrightarrow K[x_1, \dots, x_n] \to S = K[x_1, \dots, x_n]/I$$

is a Noether normalization such that S is a free R-module (of finite rank).

Exercise 3.
Let M be an R-module. Prove that the maximal elements with respect to inclusion of the family

$$\{Ann(m) \mid m \in M\}$$

of annihilator ideals $Ann(m) := \{r \in R \mid rm = 0 \in M\}$ are prime ideals.

Exercise 4

Let M be an R-module. An associated prime $\mathfrak{p}$ of M is a prime ideal of the form $\mathfrak{p} = Ann(m)$ for some $m \in M \setminus \{0\}$. We denote with $Ass(M)$ the set associated primes of M. Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of R-modules. Prove:

$$Ass(M') \subset Ass(M) \subset Ass(M') \cup Ass(M'').$$