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Fachrichtung Mathematik
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Algebraic Geometry   Summer Term 2018

Exercise Sheet 5. Hand in by Friday, May 25.

Exercise 1

Let $P = K[x_1, \ldots, x_n]$ be the polynomial ring and let $>$ be a global monomial order on the free P-module P^s of rank s.

(1) Prove the division theorem for a system $f_1, \ldots, f_r \in P^s$ of polynomial vectors.
(2) Define Gröbner basis for submodules $I \subset P^s$ and formulate Buchberger's criterion.

Exercise 2

Design an algorithm which answers - $f \in \operatorname{rad}(f_1, \ldots, f_f)$? and in case of true returns an integer N and coefficients $a_1, \ldots, a_r$ such that

$$f^N = a_1 f_1 + \ldots + a_r f_r.$$

Exercise 3

Compute the Zariski closure C of the set

$$\{(t^2+1, t(t^2+1) | t \in \mathbb{R}\} \subset \mathbb{A}^2(\mathbb{C})$$

and determine its $\mathbb{R}$-rational points $C(\mathbb{R})$.

Exercise 4

Let $g_1, g_2, h \in K[x]$ be three polynomials of degree $\leq d$, and consider the rational map

$$\varphi : \mathbb{A}^1 \dashrightarrow \mathbb{A}^2$$

defined by $(f_1, f_2) = (\frac{g_1}{h}, \frac{g_2}{h})$. Prove that the Zariski closure of the image $\varphi(U) \subset \mathbb{A}^2$ for $U = \mathbb{A}^1 \setminus V(h)$ is defined by a polynomial $F \in K[y_1, y_2]$ of degree $\leq d$ unless $\varphi(U)$ is a point.