UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 - Mathematik
Prof. Dr. Frank-Olaf Schreyer
Christian Bopp

Exercises Algebraic Geometry

Winterterm 2016/17

The solutions are collected on Monday, before the lecture.
All further informations concerning the lecture can be found here: https://www.math.uni-sb.de/ag/schreyer/index.php/teaching

Sheet 1 — 31.10.2016

Exercise 1 (1.2.2)**.** Let $\Bbbk$ be an infinite field, and let $f \in \Bbbk[x_1, \ldots, x_n]$ be a polynomial. If f is nonzero, show that the complement $\mathbb{A}^n(\Bbbk) \setminus \mathrm{V}(f)$ is an infinite set. Conclude that f is the zero polynomial iff the polynomial function $f : \mathbb{A}^n(\Bbbk) \to \Bbbk$ is zero.
Hint. Proceed by induction on the number n of variables. To begin with, recall that a nonzero polynomial in one variable has at most finitely many roots.

Exercise 2 (1.3.3)**.** Let I, I_k, J, J_k, K be ideals of R, for $1 \leq k \leq s$, and let $g \in R$. Show:

(1)
$$I : J = R \iff J \subset I.$$

(2)
$$\left(\bigcap_{k=1}^{s} I_k\right) : J = \bigcap_{k=1}^{s} (I_k : J).$$

(3)
$$I : \left(\sum_{k=1}^{s} J_k\right) = \bigcap_{k=1}^{s} (I : J_k).$$

(4)
$$(I : J) : K = I : JK.$$

(5)
$$I : g^m = I : g^{m+1} \Longrightarrow I = (I : g^m) \cap \langle I, g^m \rangle.$$

Exercise 3 (1.5.4)**.**

(1) Show that every polynomial $f \in \Bbbk[x, y, z]$ has a representation of type
$$f = g_1(y - x^2) + g_2(z - x^3) + h,$$
where $g_1, g_2 \in \Bbbk[x, y, z]$ and $h \in \Bbbk[x]$.

(2) Let $\Bbbk$ be infinite, and let $C = \mathrm{V}(y - x^2, z - x^3) \subset \mathbb{A}^3(\Bbbk)$ be the **twisted cubic curve** in $\mathbb{A}^3(\Bbbk)$. Show that
$$\mathrm{I}(C) = \langle y - x^2, z - x^3 \rangle.$$

Hint. To obtain the representation in part 1, first suppose that f is a monomial. For part 2, use that C can be parametrized:
$$C = \{(a, a^2, a^3) \mid a \in \Bbbk\}.$$

Exercise 4 (1.5.5)**.** Let $\Bbbk = \mathbb{R}$, and let
$$C = \{(a^2 + 1, a^3 + a) \mid a \in \mathbb{R}\} \subset \mathbb{A}^2(\mathbb{R}).$$
Show that $\mathrm{I}(C) = \langle y^2 - x^3 + x^2 \rangle$, and conclude that $\overline{C} = C \cup \{(0, 0)\}$.