UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 - Mathematik
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Exercises Algebraic Geometry

Winterterm 2016/17

The solutions are collected on Tuesday, before the exercise session.
All further informations concerning the lecture can be found here: https://www.math.uni-sb.de/ag/schreyer/index.php/teaching

Sheet 10 18.01.2017

Exercise 1 (5.3.12)**.** For each set of integers $r_1, \dots, r_s \geq 1$, show by example that the conclusion of the proposition may be wrong if $d = (\sum r_i) - 2$.

Exercise 2 (5.4.6)**.** Consider the polynomials

$$f = xy^2 - xy - y^3 + 1, \quad g = x^2y^2 - x^2y + xy - 1 \in \mathbb{Q}[x, y].$$

(1) Compute that

$$\begin{aligned} \operatorname{Res}(f, g, x) &= \det \begin{pmatrix} y^2 - y & 0 & y2 - y \\ -y^3 + 1 & y^2 - y & y \\ 0 & -y^3 + 1 & -1 \end{pmatrix} \\ &= y^8 - y^7 + y^6 - 3y^5 + y^4 + y^3 + y^2 - y \\ &= y(y-1)^2(y^5 + y^4 + 2y^3 - y - 1). \end{aligned}$$

Since the resultant is contained in the elimination ideal $\langle f, g \rangle \cap \mathbb{Q}[y]$, the y-values of the complex solutions of $f = g = 0$ must be among its roots. This gives eight candidates for the y-values.

(2) If $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ is projection onto the y-component, show that

$$\pi(\mathrm{V}(f, g)) \subsetneq \mathrm{V}(\operatorname{Res}(f, g)).$$

Exactly, what y-value does not have a preimage point?

(3) Use Gröbner bases to compute that the elimination ideal $\langle f, g \rangle \cap \mathbb{Q}[y]$ is generated by the polynomial $(y-1)^2(y^5 + y^4 + 2y^3 - y - 1)$. Compare this with the result of the previous part.

Exercise 3 (5.4.7)**.** Let $f, g \in \Bbbk[x_1, \dots, x_n]$ be forms of degrees $d, e \geq 1$. Suppose that both $f(1, 0, \dots, 0)$ and $g(1, 0, \dots, 0)$ are nonzero. That is, the leading coefficients of f and g – regarded as polynomials in x_1 – are nonzero constants. Then show that $\operatorname{Res}(f, g, x_1)$ is homogeneous of degree $d \cdot e$. $\square$

Exercise 4. Let $p_1, \dots, p_4 \in \mathbb{A}^2$ be the four edge points of a convex quadrilateral. Show that there is no parabola through $p_1, \dots, p_4$ if the points form a parallelogram. *Hint:* A parabola in $\mathbb{A}^2$ is the affine part of an irreducible conic in $\mathbb{P}^2$ which intersects the line at infinity in a single point with multiplicity 2.