UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 - Mathematik
Prof. Dr. Frank-Olaf Schreyer
Christian Bopp

Exercises Algebraic Geometry

Winterterm 2016/17

The solutions are collected on Tuesday, before the exercise session.
All further informations concerning the lecture can be found here: https://www.math.uni-sb.de/ag/schreyer/index.php/teaching

Sheet 2 07.11.2016

Exercise 1 (1.5.12)**.** Let I, J be ideals of a ring R. Show:

(1) $$\mathrm{rad}(IJ) = \mathrm{rad}(I \cap J) = \mathrm{rad}I \cap \mathrm{rad}J.$$

(2) $$\mathrm{rad}(I + J) = \mathrm{rad}(\mathrm{rad}I + \mathrm{rad}J).$$

(3) $$\mathrm{rad}I = \langle 1 \rangle \iff I = \langle 1 \rangle.$$

(4) If $\mathrm{rad}I, \mathrm{rad}J$ are coprime, then I, J are coprime as well. Two ideals $I, J \subset R$ are called **coprime** if $I + J = \langle 1 \rangle$.

Exercise 2 (1.6.5)**.** Let $I \subset \mathbb{k}[x_1, \dots, x_n]$ be an ideal, and let $\overline{\mathbb{k}}$ be the algebraic closure of $\mathbb{k}$. Show that the following are equivalent:

(1) The locus of zeros of I in $\mathbb{A}^n(\overline{\mathbb{k}})$ is a finite set of points (or empty).
(2) For each i, $1 \leq i \leq n$, there is a nonzero polynomial in $I \cap \mathbb{k}[x_i]$.
(3) The $\mathbb{k}$-vector space $\mathbb{k}[x_1, \dots, x_n]/I$ has finite dimension.

Exercise 3 (1.9.3)**.** Let R be a Noetherian ring, let $\mathfrak{m}$ be a maximal ideal of R, and let I be any ideal of R. Show that the following are equivalent:

(1) I is $\mathfrak{m}$-primary.
(2) $\mathrm{rad}I = \mathfrak{m}$.
(3) $\mathfrak{m} \supset I \supset \mathfrak{m}^k$ for some $k \geq 1$.

Exercise 4 (2.1.2 Gordan's Lemma)**.** By induction on the number of variables, show that any nonempty set X of monomials in $\mathbb{k}[x_1, \dots, x_n]$ has only finitely many minimal elements in the partial order given by divisibility ($x^\alpha \geq x^\beta$ iff $\alpha - \beta \in \mathbb{N}^n$). Conclude that any monomial ideal of $\mathbb{k}[x_1, \dots, x_n]$ has finitely many monomial generators.