UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 - Mathematik
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Exercises Algebraic Geometry

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The solutions are collected on Tuesday, before the exercise session.
All further informations concerning the lecture can be found here: https://www.math.uni-sb.de/ag/schreyer/index.php/teaching

Sheet 3 14.11.2016

Exercise 1 (2.2.11)**.** Let $>$ be a monomial order on $\Bbbk[x_1,\dots,x_n]$, and let X be a finite set of monomials in $\Bbbk[x_1,\dots,x_n]$. Prove that there exists a weight order $>_w$ on $\Bbbk[x_1,\dots,x_n]$ which coincides on X with the given order $>$. If $>$ is global, show that $>_w$ can be chosen to be global as well.
Hint. Consider the set of differences $\{\alpha-\beta \mid x^\alpha, x^\beta \in X, x^\alpha > x^\beta\}$, and show that its convex hull in $\mathbb{R}^n$ does not contain the origin. For the second statement, add $1, x_1, \dots, x_n$ to X if necessary.

Exercise 2 (2.2.15)**.** Define a global monomial order on $\Bbbk[x,y,z]$ yielding the leading terms y of $y-x^2$ and z of $z-x^3$, and reconsider part 1 of Exercise 1.5.4.

Remark (2.2.20). One way of getting a monomial order on F is to pick a monomial order $>$ on R, and extend it to F. For instance, setting

$$x^\alpha e_i > x^\beta e_j \iff x^\alpha > x^\beta \text{ or } (x^\alpha = x^\beta \text{ and } i > j)$$

gives priority to the monomials in R, whereas the order defined below gives priority to the components of F:

$$x^\alpha e_i > x^\beta e_j \iff i > j \text{ or } (i = j \text{ and } x^\alpha > x^\beta).$$

Exercise 3 (2.2.22)**.** Consider $F = \Bbbk[x,y]^3$ with its canonical basis and the vectors

$$g = \begin{pmatrix} x^2y + x^2 + xy^2 + xy \\ xy^2 - 1 \\ xy + y^2 \end{pmatrix},\ f_1 = \begin{pmatrix} xy + x \\ 0 \\ y \end{pmatrix},\ f_2 = \begin{pmatrix} 0 \\ y^2 \\ x+1 \end{pmatrix} \in F.$$

Extend $>_{\text{lex}}$ on $\Bbbk[x,y]$ to F in the two ways described in Remark 2.2.20. With respect to both orders, find $\mathbf{L}(g)$, $\mathbf{L}(f_1)$, and $\mathbf{L}(f_2)$, and divide g by f_1 and f_2 (use the determinate division algorithm).

Remark-Definition (2.3.6). In the situation of Macaulay's theorem, given $g \in F$, the remainder h in a standard expression $g = \sum_{i=1}^r g_i f_i + h$ satisfying (DD2) is uniquely determined by g, I, and $>$ (and does not depend on the choice of Gröbner basis). It represents the residue class $g + I \in F/I$ in terms of the standard monomials (the monomials not contained in $\mathbf{L}_>(I)$). We write $\mathrm{NF}(g,I) = h$ and call $\mathrm{NF}(g,I)$ the **canonical representative** of $g + I \in F/I$ (or the **normal form** of g mod I), with respect to $>$.

Exercise 4 (2.3.7)**.** Let $I \subset \Bbbk[x_1,\dots,x_n]$ be an ideal. If $f, g \in \Bbbk[x_1,\dots,x_n]$, show that

$$\mathrm{NF}(f+g, I) = \mathrm{NF}(f,I) + \mathrm{NF}(g,I), \text{ and}$$
$$\mathrm{NF}(f\cdot g, I) = \mathrm{NF}(\mathrm{NF}(f,I)\cdot \mathrm{NF}(g,I), I).$$