UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 - Mathematik
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Exercises Algebraic Geometry

Winterterm 2016/17

The solutions are collected on Tuesday, before the exercise session.
All further informations concerning the lecture can be found here: https://www.math.uni-sb.de/ag/schreyer/index.php/teaching

Sheet 4 — 21.11.2016

Definition 1. *A monomial order* $>$ *on the polynomial ring* $\Bbbk[\mathbf{x},\mathbf{y}] = \Bbbk[x_1,\dots,x_n,y_1,\dots y_m]$ *is an* **elimination order** *with respect to* $x_1,\dots,x_n$ *if the following holds for all* $f \in \Bbbk[\mathbf{x},\mathbf{y}]$*:* $\mathbf{L}(f) \in \Bbbk[\mathbf{y}] \implies f \in \Bbbk[\mathbf{y}]$.

Exercise 1 (2.5.13)**.** [Subalgebra Membership] With notation as above, let $\overline{g}, \overline{f}_1,\dots,\overline{f}_m$ be elements $\Bbbk[x_1,\dots,x_n]/I$, and let $>$ be a global elimination order on $\Bbbk[\mathbf{x},\mathbf{y}]$ with respect to $x_1,\dots,x_n$. Show:

(1) We have $\overline{g} \in \Bbbk[\overline{f}_1,\dots,\overline{f}_m]$ iff the normal form $h = \mathrm{NF}(g,J) \in \Bbbk[\mathbf{x},\mathbf{y}]$ is contained in $\Bbbk[\mathbf{y}]$. In this case, $\overline{g} = h(\overline{f}_1,\dots,\overline{f}_m)$ is a polynomial expression for $\overline{g}$ in terms of the $\overline{f}_k$.
(2) The homomorphism $\phi : \Bbbk[y_1,\dots,y_m] \to \Bbbk[x_1,\dots,x_n]/I$ is surjective iff $\mathrm{NF}(x_i,J) \in \Bbbk[\mathbf{y}]$ for $i = 1,\dots,n$. $\square$

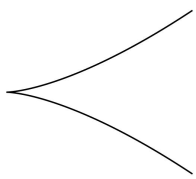
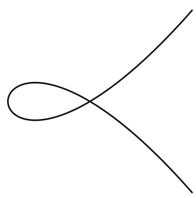
Exercise 2 (2.6.19)**.** Let $I \subset \Bbbk[x_1,\dots,x_n]$ be a prime ideal, let $V = \mathrm{V}(I) \subset \mathbb{A}^n(\Bbbk)$ be the corresponding variety, and let $\varphi : V \dashrightarrow \mathbb{A}^m(\Bbbk)$ be a rational map given by rational functions $f_i = (g_i + I)/(h_i + I) \in \Bbbk(V)$, where the $g_i, h_i \in \Bbbk[x_1,\dots,x_n]$. Supposing that V is Zariski dense in the locus of zeros of I in $\mathbb{A}^n(\overline{\Bbbk})$, design an algorithm which computes the Zariski closure of $\varphi(\mathrm{domain}(\varphi)) \subset \mathbb{A}^m(\Bbbk)$.

Exercise 3 (2.2.22)**.** Consider the polynomial parametrizations

$$\mathbb{A}^1(\Bbbk) \to \mathrm{V}(y^2 - x^3) \subset \mathbb{A}^2(\Bbbk),\ a \mapsto (a^2, a^3),$$

and

$$\mathbb{A}^1(\Bbbk) \to \mathrm{V}(y^2 - x^3 - x^2) \subset \mathbb{A}^2(\Bbbk),\ a \mapsto (a^2 - 1, a^3 - a).$$

Show that each of the parametrizations admits a rational inverse. Use these examples to show that the domain of definition of the composite $\psi \circ \varphi$ of two rational maps may be strictly larger than $\varphi^{-1}(\mathrm{domain}(\psi))$.

Exercise 4 (2.3.7)**.** If $\mathrm{char}\Bbbk \neq 2,3$, find a rational parametrization of the affine plane curve with equation $y^3 - 3x^2y = (x^2 + y^2)^2$: