UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 - Mathematik
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Exercises Algebraic Geometry

Winterterm 2016/17

The solutions are collected on Tuesday, before the exercise session.
All further informations concerning the lecture can be found here: https://www.math.uni-sb.de/ag/schreyer/index.php/teaching

Sheet 5 28.11.2016

Exercise 1 (3.1.5)**.** Consider the ideal

$$I = \langle xy(x+y)+1 \rangle \subset \mathbb{F}_2[x,y].$$

Determine coordinates in which I satisfies the extra hypothesis of the projection theorem. Show that the extra hypothesis cannot be achieved by means of a *linear* change of coordinates.

Exercise 2 (3.1.8)**.** Check that the polynomials

$$f_1 = x^3 - xz, \quad f_2 = yx^2 - yz \in \Bbbk[x,y,z]$$

form a lexicographic Gröbner basis. Conclude that $\mathrm{V}(f_1, f_2) \subset \mathbb{A}^3$ projects *onto* the yz-plane. Determine the points of the yz-plane with 1,2, and 3 preimage points, respectively.

Exercise 3 ((3.2.6) - Integrality Criterion for Affine Rings)**.**
Let I be an ideal of $\Bbbk[x_1,\ldots,x_n]$, and let $\overline{f}_1 = f_1 + I, \ldots, \overline{f}_m = f_m + I \in \Bbbk[x_1,\ldots,x_n]/I$. Consider a polynomial ring $\Bbbk[y_1,\ldots,y_m]$, the homomorphism

$$\phi : \Bbbk[y_1,\ldots,y_m] \to S = \Bbbk[x_1,\ldots,x_n]/I, \; y_i \mapsto \overline{f}_i,$$

and the ideal

$$J = I\,\Bbbk[\mathbf{x},\mathbf{y}] + \langle f_1 - y_1, \ldots, f_m - y_m \rangle \subset \Bbbk[\mathbf{x},\mathbf{y}].$$

Let $>$ be an elimination order on $\Bbbk[\mathbf{x},\mathbf{y}]$ with respect to $x_1,\ldots,x_n$, and let $\mathcal{G}$ be a Gröbner basis for J with respect to $>$. By Proposition 2.5.12 the elements of $\mathcal{G} \cap \Bbbk[\mathbf{y}]$ generate $\ker\phi$. View $R := \Bbbk[y_1,\ldots,y_m]/\ker\phi$ as a subring of S by means of ϕ. Show that $R \subset S$ is integral iff for each i, $1 \le i \le n$, there is an element of $\mathcal{G}$ whose leading monomial is of type $x_i^{\alpha_i}$ for some $\alpha_i \ge 1$.

Exercise 4 (3.2.11)**.** If R is a ring, show that its nilradical is the intersection of all the prime ideals of R:

$$\mathrm{rad}\langle 0 \rangle = \bigcap_{\mathfrak{p} \subset R \text{ prime}} \mathfrak{p}.$$