UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 - Mathematik
Prof. Dr. Frank-Olaf Schreyer
Christian Bopp

Exercises Algebraic Geometry

Winterterm 2016/17

The solutions are collected on Tuesday, before the exercise session.
All further informations concerning the lecture can be found here: https://www.math.uni-sb.de/ag/schreyer/index.php/teaching

Sheet 7 12.12.2016

Exercise 1.

(a) (4.2.2) [Universal Property of Localization]
Let R be a ring, and let $U \subset R$ be a multiplicatively closed subset. Show that if $\phi : R \to S$ is a homomorphism of rings which maps the elements of U to units, there exists a uniquely determined homomorphism $\Phi : R[U^{-1}] \to S$ such that the diagram

$$\begin{tikzcd} R \arrow[rr, "\phi"] \arrow[dr, "\iota"'] & & S \\ & R[U^{-1}] \arrow[ur, dashed, "\Phi"'] & \end{tikzcd}$$

commutes.

(b) (4.2.3) [Localization Commutes with Passing to Quotients by Ideals]
Let R and U be as above, let $I \subset R$ be an ideal, and let $\overline{U}$ be the image of U in R/I. Then show that the natural map

$$R \to R[U^{-1}] \to R[U^{-1}]/I\,R[U^{-1}]$$

induces an isomorphism

$$(R/I)[\overline{U}^{-1}] \cong R[U^{-1}]/I\,R[U^{-1}].$$

Exercise 2.

(a) (4.2.8) [Localization Commutes with Forming Radicals]
If $I \subset R$ is an ideal, then show that $\mathrm{rad}(IR[U^{-1}]) = (\mathrm{rad}I)R[U^{-1}]$. Conclude that the injection $J \mapsto \iota^{-1}(J)$ restricts to a bijection between the set of primary ideals of $R[U^{-1}]$ and the set of primary ideals of R not meeting U.

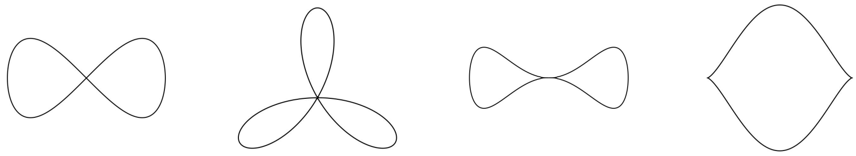
(b) Let R be a ring and $f \in R$. Show:

$$R_f = 0 \Leftrightarrow f^n = 0 \text{ for some } n \in \mathbb{N}$$

Exercise 3 (4.2.22)**.** (1) Find an ideal of $\Bbbk[x_1, \dots, x_n]$ which admits minimal sets of generators differing in their number of elements.

(2) Let $\mathcal{O}_{\mathbb{A}^2,o}$ be the local ring of $\mathbb{A}^2$ at the origin $o = (0, 0)$. For each $n \in \mathbb{N}$, find an ideal of $\mathcal{O}_{\mathbb{A}^2,o}$ which is minimally generated by n elements.

Exercise 4 (4.3.6)**.** The following picture shows plane curves with different types of singularities:



node triple point tacnode cusps

The curves above are defined by the polynomials below:

$$y^2 = (1-x^2)^3, \quad y^2 = x^2 - x^4, \quad y^3 - 3x^2y = (x^2+y^2)^2, \quad y^2 = x^4 - x^6.$$

Which curve corresponds to which polynomial?