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Intrinsic Ultracontractivity and Uniform Conditional Ergodicity

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Preface

This *Diplom* thesis was written at the *Institut für Mathematische Stochastik* at the *Technische Universität Dresden* under the supervision of Dr. Lothar Partzsch and Professor Rolf Kühne.

This thesis deals with certain properties of multidimensional diffusion semigroups. We introduce some underlying concepts of multidimensional diffusion processes, and we develop some new results related to this field. Our main result states that under suitable assumptions on the diffusion and drift coefficients of a non–symmetric second–order differential operator in nondivergence form the associated diffusion semigroup is intrinsically ultracontractive on a $C^{2,1}$–domain. Furthermore, we give a stochastic motivation for dealing with intrinsic ultracontractivity by showing that as a consequence of that property the corresponding diffusion semigroup is uniformly conditionally ergodic and has a unique quasi–stationary distribution.

What sets this work apart from the literature is that criteria as well as equivalent characterisations for intrinsic ultracontractivity of non–symmetric diffusion semigroups are developed without relying on the results concerning intrinsic ultracontractivity for symmetric diffusions. So far, it seems that the interrelation between intrinsic ultracontractivity and uniform conditional ergodicity has not yet been studied, and thus the present thesis adds yet another example of the prolific connection between analysis and stochastics.

I would like to use this occassion to express my deepest gratitude towards my thesis supervisor, Dr. Lothar Partzsch, for his constant support and valuable ideas, especially those related to the stochastic interpretation of intrinsic ultracontractivity, without which the thesis in the present form would not have been possible. In particular, his extraordinary commitment deserves to be mentioned here. He was always available when I needed his advice, indeed he was a true partner in the elaboration on this thesis. I really enjoyed this collaboration, and I'm also deeply grateful for the many informative discussions, which provided me with a vast number of important insights, many of them do not appear in this thesis but have been indispensable in the process of developing it.

Robert Knobloch
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Contents

Introduction

The aim of this thesis is to study certain analytical and ergodic properties of multidimensional diffusion processes, which constitute a particular class of Markov processes. Markov processes and diffusion processes are well studied objects in mathematical stochastics, but yet there are many things that require clarification. To begin with, the question arises "What is a diffusion process?" Well, most people working in that field will have an answer to this question, but these answers may differ. Analysts may construct diffusion processes via semigroup theory and utilising the Hille–Yosida theorem as well as the theory of partial differential equations, or they may take the approach via Dirichlet forms. Probabilists may use for instance the approach by Dynkin (cf. 5.26 in [Dyn65I]) or the approach via stochastic differential equations, or they may apply the martingale problem. Even though all these approaches lead to similar objects, it may often be difficult to see the similarities. We therefore give two approaches to construct diffusion processes. Firstly, in order to properly define these processes for our further proceeding, and secondly in order to show the similarities and the differences between the two approaches.

The consideration of problems related to the questions with which we are concerned has quite a long history. In his dissertation (cf. [Par72]) the supervisor of this thesis, L. Partzsch, was, among other things, dealing with similar questions in the one–dimensional case. For some of the problems considered in [Par72] we present an approach in the multidimensional situation, and in some ways relations between results in [Par72] and the results in the present thesis become obvious, even though the methods in [Par72] differ profoundly from our approach.

For some problems similar to the underlying questions of the present thesis solutions can be found in the literature, where the theory of multidimensional diffusions has been studied extensively.

We consider some bounded domain $U \subseteq \mathbb{R}^d$ and an operator $T : C_K^2(U) \to C_K(U)$ in nondivergence form given by

$$\forall\, f \in C_K^2(U) : Tf = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} f \,. \tag{1}$$

Moreover, assume that for all $x \in U$ and $t > 0$ the corresponding transition measure possesses a positive λ_d–density $p_t(x, \cdot)$, and that for T and its formal adjoint T^* the normalised eigenfunctions φ and ψ corresponding to the principal eigenvalue are positive. The stochastically motivated question with which we are concerned is related to a convergence property of diffusion semigroups, namely the so–called uniform conditional ergodicity. The diffusion semigroup $(P_t)_{t\in\mathbb{R}_0^+}$ associated with T is referred to as uniformly conditionally ergodic if there exists a probability measure μ on $(U, \mathscr{B}(U))$ such that

$$\frac{P_t(x,B)}{P_t(x,U)} \to \mu(B) \text{ uniformly in } x \in U \text{ and } B \in \mathscr{B}(U) \text{ as } t \to \infty.$$

Here $\mathscr{B}(U)$ denotes the Borel σ–algebra on U.

One purpose of this thesis is to give criteria such that a diffusion semigroup is uniformly conditionally ergodic. Our method to tackle this problem is based upon the notion of intrinisic ultracontractivity, which is one of the basic concepts of this thesis. We give conditions on the coefficients of T, given by (1), such that the corresponding diffusion semigroup is intrinsically ultracontractive, and we show that intrinisic ultracontractivity implies uniform conditional ergodicity. In fact, the latter point motivates our interest in intrinisic ultracontractivity.

Intrinsic ultracontractivity is a primarily functional analytical property, which at first sight seems to have little relation to our problem, since its original definition (cf. Section 3 in [DS84]) bears little obvious relation to the situation which we want to study. However, that relation becomes more obvious if one realises that the original definition, in the context of symmetric diffusions, is equivalent to the following property:

$$\forall\, t > 0\, \exists\, \alpha_t > 0\, \forall\, x, y \in U : \frac{p_t(x,y)}{\varphi(x)\varphi(y)} \leq \alpha_t,$$

where p_t and $\varphi = \psi$ are as above. We use the non–symmetric analogue of the above property as the definition of intrinsic ultracontractivity. The interpretation of intrinsic ultracontractivity certainly depends on the point of view, and an analyst will probably see it from a different perspective than a probabilist. For interpretations and equivalent characterisations of intrinsic ultracontractivity see Chapter 3. Our main work is to find criteria under which that property holds. The purpose being to examine when diffusion semigroups are intrinsically ultracontractive, in order to derive conclusion relating to the ergodic behaviour of these diffusions. In the literature most authors have concerned themselves with intrinsic ultracontractivity in the context of symmetric diffusions, and by now that context has been studied quite extensively. In contrast, in the context of non–symmetric diffusions astonishingly only very little has been done so far. In fact, with the exception of [KS06a] by Kim and Song it seems that this situation has not been studied at all. A possible reason may be that it is easier to deal with intrinsic ultracontractivity in the symmetric case. If one has a self–adjoint operator which admits a series expansion via its eigenvalues and corresponding normalised eigenfunctions, then one obtains quite nice criteria for intrinsic ultracontractivity to hold. Moreover, having a symmetric Lebesgue density for the transition measure turns out to be of great avail.

Well, in the non–symmetric situation we generally do not have any of these properties at hand, and the criteria for intrinsic ultracontractivity in the symmetric case do not work in the non–symmetric situation. The only paper in respect of intrinsic ultracontractivity of non–symmetric diffusion semigroups seems to be [KS06a], which gives conditions on the coefficients of a second–order differential operator such that the corresponding diffusion semigroup is intrinsically ultracontractive. The method used by Kim and Song in [KS06a] is to utilise that under certain assumptions on the diffusion and drift coefficients of the generator of a symmetric diffusion the corresponding diffusion semigroup is intrinsically ultracontractive on a bounded Lipschitz domain. They were concerned with non–symmetric diffusion semigroups in bounded Lipschitz domains, whose coefficients satisfy sufficiently strong conditions. The idea by Kim and Song is to consider the symmetric semigroup corresponding to the arithmetic average of the generator of a non-symmetric diffusion and its formal adjoint.

However, in the present thesis we do not take the same approach for two reasons. Firstly, Kim and Song considered the Hilbert space context, i.e., their transition operators are defined on $\mathscr{L}^2$, whereas our transition operators are defined on C_0. Moreover, their method rests on utilising Dirichlet forms and logarithmic Sobolev inequalities. Our aim is to develop a more stochastic approach by utilising the martingale problem. Secondly, the proceeding in [KS06a] relies on the fact that the corresponding symmetric diffusions are intrinsically ultracontractive on bounded Lipschitz domains, whereas our aim is to develop an integrated concept from scratch of showing that intrinsic ultracontractivity holds. Our criteria for non–symmetric diffusion semigroups to be intrinsically ultracontractive are presented in Chapter 3.

Now we would like to give a brief outline of the particular chapters. This thesis is divided into two parts. Our intention for the first part, which comprises Chapter 1 and Chapter 2, is to introduce the notion of multidimensional diffusion processes in a domain in $\mathbb{R}^d$ and to present ways to construct them, as well as to derive a few properties of these processes. These diffusions are the main objects of our interest in Part II, which consists of Chapter 3 and Chapter 4. There we study certain analytical and ergodic properties of the associated diffusion semigroups. Therefore, Part I can be understood as a kind of preparation for the second part. Part II is the main part of this thesis, and in this part we develop and present our main results. However, even though the intention for Part I is to provide the background for Part II, in particular the results in Chapter 2 are more comprehensive than necessary for that purpose. In fact, we include the approach to obtain diffusions via the martingale problem, because it may also provide a basis for dealing with further questions which are related to the problem that we consider here, but are not covered by the present thesis.

In the first section of Chapter 1 we introduce the Markov property, and we are concerned with some general theory related to Markov processes with values in some locally compact Polish space. That section is about canonical Markov processes generated via Kolmogorov's extension theorem by starting with a Markov semigroup of transition kernels. The resulting class of canonical Markov processes is far too large to be useful, since the processes do not need to have any regularity properties, i.e., there are no constraints regarding continuity of the trajectories. This shortcoming is partly evaded in Section 1.2, which restricts the processes under consideration to Feller processes. Roughly speaking, Feller processes are canonical Markov processes whose corresponding semigroup of transition operators is a Feller semigroup. In particular, Feller processes are right–continuous processes with left–hand limits (so–called rcll processes). In Section 1.3 we give a first definition of diffusion processes, for that we introduce the notion of Feller diffusions in $\mathbb{R}^d$. A Feller diffusion in $\mathbb{R}^d$ is a continuous $\mathbb{R}^d$–valued Feller process. Subsequently, we consider a domain $U \subseteq \mathbb{R}^d$ and the corresponding process which is killed upon reaching the boundary of U. This killed process is referred to as a Feller diffusion in U. The construction of such Feller diffusions in U is the main purpose of Chapter 1. Our considerations concerning Feller processes are mainly based upon [Kal01] by Kallenberg.

Chapter 2 is devoted to pretty much the same intention as Chapter 1, namely to clarify the notion of a diffusion process in a domain in $\mathbb{R}^d$. The approach in this chapter is very different from the one in Chapter 1, but nonetheless it turns out that the resulting diffusions have many basic properties in common with the Feller diffusions as constructed in Chapter 1. In Section 2.1 we are concerned with the martingale problem on $\mathbb{R}^d$. This section intends to provide the basic ideas for Section 2.2, in which we consider a localisation of the theory compiled in Section 2.1 to some domain $U \subseteq \mathbb{R}^d$. The purpose of Section 2.2 is to obtain a unique family of probability measures associated with a certain second–order differential operator on $C_K^2(U)$. This family of probability measures then defines a diffusion process with values in some domain $U \subseteq \mathbb{R}^d$. The uniqueness of the solution yields that the family of probability measures satisfies the Feller property as well as the strong Markov property. We give the definition of diffusions as well as a discussion about its relation to Feller diffusions in U in Section 2.3. The concept of the martingale problem was developed by Stroock and Varadhan (cf. [SV79]). Moreover, our proceeding is strongly influenced by [Pin95], where Pinsky develops a generalisation of the martingale problem by Stroock and Varadhan. The results of this chapter are not entirely exploited in the present thesis. One advantage of the martingale problem over the approach in Chapter 1 is that the boundedness assumption on the coefficients can be relaxed to require locally bounded coefficients. This advantage becomes important if one considers diffusions conditional on not leaving U, because the drift terms of those conditional processes, which may also be referred to as Doob's h–processes, are locally bounded, but not bounded. These conditional processes are also diffusions in U, and thus one can take full advantage of what is known about diffusions when dealing with conditional processes. As mentioned before, we do not consider such conditional processes in this thesis, but Chapter 2 provides the basis to deal with these processes in the context of diffusion processes. This may be an interesting starting point for further considerations based on our work, because these conditional diffusions are closely realted to the concepts with which we are concerned in the present thesis. Another virtue of the approach via the martingale problem is that the diffusion matrix does not have to be uniformly elliptic.

Recapitulating, in Part I we present two different ways of constructing multidimensional diffusion processes in a domain in $\mathbb{R}^d$. In Part II we study some properties of those diffusions, but the first part has a meaning in its own right, because, idependent of the second part, it can be considered as an introduction to the notion of multidimensional diffusion processes. Therefore, it may be justified that we consider those diffusions in more generality than we need for Part II. Now let us explain what we do in Part II.

In Chapter 3 we consider a $C^{2,1}$–domain $U \subseteq \mathbb{R}^d$ and a diffusion corresponding to a second–order differential operator T, where we impose some pretty restrictive conditions on the coefficients of T. Almost all of these conditions on the coefficients as well as the restriction on the domain turn out to be necessary for our considerations in Section 3.1, where we derive the basic properties of the diffusions corresponding to T. In particular the results regarding a Lebesgue density for the transition measure as well as spectral theoretical considerations are indispensable for our further proceeding. In this section we establish the main tools which we utilise in the course of our considerations. In particular, we show that under our conditions on the coefficients we have a positive continuous transition density with respect to the Lebesgue measure, and that for T and its formal adjoint T^* the normalised eigenfunctions corresponding to the principal eigenvalue are positive. Section 3.2 is devoted to the main concept of the whole chapter, namely intrinsic ultracontractivity. The notion of intrinsic ultracontractivity for symmetric processes was introduced by Davies and Simon in [DS84]. They considered a functional analytical approach, as have done most authors dealing with that property. Most papers relating to that property deal with the question under which assumptions intrinsic ultracontractivity holds. Yet, there are several approaches to that problem. The most common approach seems to be that authors consider a "nice" diffusion, and then they try to find criteria on how "bad" the underlying domain can be, i.e., how irregular its boundary can be, such that the diffusion in this domain is intrinsically ultracontractive. Another approach, which was considered by Cipriani (cf. [Cip94]) as well as by Ouhabaz and Wang (cf. [OW07]) is to examine under which assumptions on the normalised eigenfunction, corresponding to the principal eigenvalue, the diffusion semigroup is intrinsically ultracontractive. A third approach, which was employed by Kim and Song in [KS06a], is to fix a domain and to try to find conditions on the coefficients of the generating operator such that intrinsic ultracontractivity holds. For the present thesis we have chosen the latter approach. As mentioned above, almost exclusively self–adjoint operators have been studied in the literature. The only exception which we managed to find is [KS06a] by Kim and Song. Although the methods employed by Kim and Song differ profoundly from our considerations, the main result and some underlying questions in [KS06a] are closely related to the purpose of Section 3.2. Bearing this in mind, we briefly compare our main result with [KS06a] at the end of Chapter 3.

In Chapter 4 we give a stochastic motivation for dealing with intrinsic ultracontractivity. We show that intrinsic ultracontractivity implies that the corresponding diffusion semigroup is uniformly conditionally ergodic, which results in a unique quasi–stationary distribution. This is not obvious by the definition of intrinsic ultracontractivity, and we apply results about intrinsic ultracontractivity which we have developed in Chapter 3. A further interesting question may be whether the converse holds true, i.e., whether uniform conditional ergodicity implies intrinsic ultracontractivity. As far as we know this is still an open problem. In our opinion it seems to be likely that the converse does not hold true, because uniform conditional ergodicity does not give any information about "small" $t > 0$, whereas intrinsic ultracontractivity is a property that concerns all $t > 0$. To the best of our knowledge the interrelation between intrinsic ultracontractivity and uniform conditional ergodicity has not yet been considered in the literature.

In conclusion we would like to point out that some definitions as well as some results, which are well known or have only minor relevance for our considerations, are left for the appendices. This applies in particular to some Markov process theory, which aims to provide a background for Chapter 1.

List of Notations

Numbers

$\mathbb{N}$	natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Q}$	rational numbers
$\mathbb{Q}_0^+$	$\{x \in \mathbb{Q} : x \geq 0\}$
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	$\{x \in \mathbb{R} : x > 0\}$
$\mathbb{R}_0^+$	$\{x \in \mathbb{R} : x \geq 0\}$
$\bar{\mathbb{R}}_0^+$	$\mathbb{R}_0^+ \cup \{\infty\}$

Set Systems

$\mathcal{A} \vee \mathcal{G}$	$\sigma(\mathcal{A} \cup \mathcal{G})$
$\bigvee_{n\in\mathbb{N}} \mathcal{A}_n$	$\sigma(\bigcup_{n\in\mathbb{N}} \mathcal{A}_n)$
$\mathscr{B}(E)$, $(E, \mathcal{T})$ topol. space	$\sigma(\mathcal{T})$, Borel σ–algebra on E
$\mathscr{B}(E)^{\otimes 2}$	$\mathscr{B}(E) \otimes \mathscr{B}(E)$
$\mathrm{ca}(\mathscr{B}(B))$	set of all finite signed measures on $\mathscr{B}(E)$
$\delta(\mathcal{A})$	Dynkin system generated by $\mathcal{A}$
$\sigma(\mathcal{A})$	σ–algebra generated by $\mathcal{A}$
$\sigma(X)$	$\bigcup_{A\in\mathscr{A}} X^{-1}(A)$
$\sigma(X_1, \ldots, X_n)$	$\bigvee_{n\in\mathbb{N}} \sigma(X_n)$
$\mathfrak{P}(E)$	power set of E
$E^{\mathbb{R}_0^+}$	set of all maps $f : \mathbb{R}_0^+ \to E$
$\mathscr{Z}(E^{\mathbb{R}_0^+})$	σ–algebra generated by the cylinder sets in $E^{\mathbb{R}_0^+}$

Probability Theory

$(\Omega, \mathscr{F})$ measurable space,
$\mathscr{G}$ filtration on $(\Omega, \mathscr{F})$:

$\mathcal{P}(\Omega)$	set of all probability measures on $(\Omega, \mathscr{F})$
$\mathcal{S}(\mathscr{G})$	set of all $\mathscr{G}$–stopping times
$\mathcal{S}_b(\mathscr{G})$	set of all bounded $\mathscr{G}$–stopping times

$\mathcal{S}_f(P,\mathscr{G})$, $P \in \mathcal{P}(\Omega)$	set of all P–a.s. finite $\mathscr{G}$–stopping times
τ_U	$\inf\{t \in \mathbb{R}_0^+ : X_t \notin U\}$

Functions and Measures

$\mathbf{0}$	function identical 0
$\mathbf{1}$	function identical 1
$\mathbb{1}$	indicator function
δ_x	Dirac measure, i.e., $\delta_x(B) = \mathbb{1}_B(x)$
$\mathcal{B}(E,\mathscr{E})$, $\mathscr{E}$ σ–algebra	$\{f : E_1 \to \mathbb{R} : f$ is $\mathscr{E}$–$\mathscr{B}(\mathbb{R})$–measurable, bounded$\}$
$\mathcal{B}(E)$	$\mathcal{B}(E,\mathscr{B}(E))$
$C^n(E_1,E_2)$, $n \in \bar{N}_0$	$\{f : E_1 \to E_2 : f$ n–times continuously differentiable
$C_b^n(E_1,E_2)$, $n \in \bar{N}_0$	$\{f \in C^n(E_1,E_2) : f$ is bounded$\}$
$C_b(E_1,E_2)$	$C_b^0(E_1,E_2)$
$C_b^n(E_1)$	$C_b^n(E_1,\mathbb{R})$
$C_K^n(E_1,E_2)$, $n \in \bar{N}_0$	$\{f \in C^n(E_1,E_2) : f$ has compact support$\}$
$C_K(E_1,E_2)$	$C_K^0(E_1,E_2)$
$C_K^n(E_1)$	$C_K^n(E_1,\mathbb{R})$
$C_0(U)$, $U \subseteq \mathbb{R}^d$	$\{f \in C_b(U) : f(x) \to 0$ as $x \to \partial U\}$
$\mathcal{B}$, C_b^n, C_0, C_K^n	endowed with the supremum norm $\|\cdot\|_\infty$
rcll	right-continuous with left–hand limits
$\mathrm{RCLL}(E_1,E_2)$	$\{f : E_1 \to E_2 :$ f is rcll$\}$
$\mathrm{RCLL}(E_1)$	$\mathrm{RCLL}(E_1,\mathbb{R})$
$(\Omega,\mathscr{F},P)$, probability space:	
$\mathscr{L}^0(\Omega)$	$\{f : \Omega \to \mathbb{R}^d : f$ is $\mathscr{F}$–$\mathscr{B}(\mathbb{R}^d)$–measurable$\}$
$\mathscr{L}^p(\Omega,P)$, $p \in \mathbb{N}$	$\{f : \Omega \to \mathbb{R} : f$ is p–integrable$\}$
$\mathscr{L}^p(\Omega)$	$\mathscr{L}^p(\Omega,P)$
$\mathcal{M}(\Omega)$	set of all measures on $(\Omega,\mathscr{F})$
$\mathcal{M}_f(\Omega)$	set of all finite measures on $(\Omega,\mathscr{F})$
$f \wedge g$	pointwise minimum of f and g
$f \vee g$	pointwise maximum of f and g
f^+	$f \vee \mathbf{0}$
f^-	$(-f) \vee \mathbf{0}$
$f \leq \alpha$	$\forall\, x \in A : f(x) \leq \alpha$
$f \leq g$	$\forall\, x \in A : f(x) \leq g(x)$
id	identity operator
λ_d	d–dimensional Lebesgue measure
λ	λ_1

$\mathrm{supp}\,(f)$	support of f

Miscellaneous

$\sum_{n\in\mathbb{N}}$	$\sum_{n=1}^{\infty}$
$\stackrel{\mathrm{w}}{\to}$	weak convergence
$\stackrel{P}{\to},\ P\in\mathcal{P}(\Omega)$	convergence in probability
$\lim_{t\to t_0+0}$	right–hand limit
$\lim_{t\to t_0-0}$	left–hand limit
$\subseteq$	subset
$\subsetneq$	strict subset
$\supseteq$	superset
$B(x,\varepsilon)$	$\{y\in E:\rho(x,y)<\varepsilon\}$, (E,ρ) metric space
$B[x,\varepsilon]$	$\{y\in E:\rho(x,y)\le\varepsilon\}$
δ_{ij}	Kronecker symbol
DCT	Dominated Convergence Theorem
$\mathcal{H}(\mathbb{R}_0^+)$	$\{(t_1,\ldots,t_n):n\in\mathbb{N},t_1<\ldots<t_n\in\mathbb{R}_0^+\}$
P–a.a.	P–almost all
P–a.s.	P–almost surely
∂U	boundary of U
$\bar{U}$	$U\cup\partial U,$ closure of U

Part I

Diffusion Processes

Chapter 1

Feller Diffusions

This chapter is devoted to the construction of Feller diffusions in a domain in $\mathbb{R}^d$. To begin with, we will briefly introduce some concepts and notions related to Markov process theory. The theory of Markov processes is relevant, because diffusion processes, which are the main mathematical objects in the present thesis, are examples of Markov processes. In Section 1.1 we will be concerned with canonical Markov processes corresponding to a Markov semigroup, i.e., with processes obtained by Kolmogorov's extension theorem. Later on, in Section 1.2, we will introduce Feller processes, which roughly speaking are rcll Markov processes associated with a Feller semigroup of transition operators. Finally, in Section 1.3 we will define Feller diffusions. At first we will construct Feller diffusions in $\mathbb{R}^d$, which are continuous $\mathbb{R}^d$–valued Feller processes. However, we are interested in diffusions in a domain $U \subseteq \mathbb{R}^d$, and hence we will consider the killed processes corresponding to Feller diffusions in $\mathbb{R}^d$, killed upon reaching ∂U. These killed processes will be called Feller diffusions in U.

In order to keep this chapter reasonably concise we will focus on the main ideas with respect to the notion of Feller diffusions, and some well known Markov process theory is left for Appendix B. Occasionally this may be inconvenient, but we believe that it enables the reader to proceed more quickly to the relevant concepts of this thesis.

For any stochastic process $X := (X_t)_{t\in\mathbb{R}_0^+}$ on some probability space $(\Omega, \mathscr{F}, P)$ we denote the finite dimensional distributions of X under P by $P_{(X_{t_1},\ldots,X_{t_n})} := P \circ (X_{t_1}, \ldots, X_{t_n})^{-1}$ for all $n \in \mathbb{N}, t_1, \ldots, t_n \in \mathbb{R}_0^+$. Moreover, let us denote the distribution of X under P by $P_X := P \circ X^{-1}$.

Let $(A, \mathscr{A})$ and $(G, \mathscr{G})$ be some measurable spaces, and let μ be a kernel on $(G, \mathscr{G})$. In addition, let $\varphi : A \to G$ be an $\mathscr{A}$–$\mathscr{G}$–measurable function and let $B \in \mathscr{G}$. Then we denote $\mu(\cdot, B) \circ \varphi$ by $\mu(\varphi, B)$. Similar terms are to be interpreted in this spirit.

Even though later on we will restrict our considerations to stochastic processes with values in $\mathbb{R}^d$ or subdomains thereof, we will start by considering more general spaces, namely Polish spaces:

Definition 1.1 *A topological space $(E, \mathcal{T})$ is called a* **Polish space** *if $\mathcal{T}$ is induced by a metric ρ such that (E, ρ) is a complete, separable metric space.*

Throughout this chapter let $(E, \mathscr{B}(E))$ be a measurable space, where $(E, \mathcal{T})$ is a locally compact Polish space and $\mathscr{B}(E) := \sigma(\mathcal{T})$. Furthermore, let ρ_E denote a metric which induces $\mathcal{T}$.

A sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(E)$, the set of all probability measures on $(E, \mathscr{B}(E))$, is said to converge **weakly** to some $\nu \in \mathcal{P}(E)$ if

$$\forall\, f \in C_b(E) : \int_E f \,\mathrm{d}\nu_n \to \int_E f \,\mathrm{d}\nu \text{ as } n \to \infty,$$

and we say that $(\nu_n)_{n\in\mathbb{N}}$ converges **vaguely** to ν if

$$\forall\, f \in C_0(E) : \int_E f \,\mathrm{d}\nu_n \to \int_E f \,\mathrm{d}\nu \text{ as } n \to \infty.$$

Since $(E, \mathcal{T})$ is a locally compact Polish space, we obtain by 4.4 Proposition in Chapter 3 in [EK86] that a sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(E)$ converges weakly to some $\nu \in \mathcal{P}(E)$ iff it converges vaguely to ν, i.e., on $\mathcal{P}(E)$ the concepts of weak convergence and vague convergence to some probability measure coincide. The weak convergence in $\mathcal{P}(E)$ induces the weak topology $\mathcal{T}_w$ on $\mathcal{P}(E)$, i.e., $\mathcal{T}_w$ is the smallest topology on $\mathcal{P}(E)$ for which the maps $\varphi_f : \mathcal{P}(E) \to \mathbb{R}$, $f \in C_b(E)$, given by $\varphi(\mu) = \int_E f \,\mathrm{d}\mu$ for all $\mu \in \mathcal{P}(E)$, are continuous. Moreover, since E is separable, $\mathcal{T}_w$ is induced by the Prohorov metric ρ_P (cf. e.g. Remark 13.14 (ii) in [Kle06] or Section 1 of Chapter 3 in [EK86]). In particular, this means that we can apply the results of Chapter 3 in [EK86], which are acquired for the metric space $(\mathcal{P}(E), \rho_P)$, to the topological space $(\mathcal{P}(E), \mathcal{T}_w)$.

Throughout the whole thesis we adopt the convention $\inf \emptyset := \infty$.

1.1 Canonical Markov Processes

To begin with, we make a few definition which introduce some basic notions relevant for our considerations.

Definition 1.2 *Let $X := (X_t)_{t\in\mathbb{R}_0^+}$ be a family of E–valued random variables on some measurable space $(\Omega, \mathscr{F})$. For any $t \in \mathbb{R}_0^+$ we define*

$$\mathscr{F}_{\geq t} := \sigma(X_s : s \in [t, \infty)\} \quad \textit{and} \quad \mathscr{F}_t := \sigma(X_s : s \in [0, t]\}$$

and obtain

$$\mathscr{F}_{\geq}^X := (\mathscr{F}_{\geq t})_{t\in\mathbb{R}_0^+} \quad \textit{and} \quad \mathscr{F}^X := (\mathscr{F}_t)_{t\in\mathbb{R}_0^+},$$

where the latter is called the **natural filtration** *with respect to X.*

Definition 1.3 *Let $(X_t)_{t\in\mathbb{R}_0^+}$ be a family of E–valued random variables on some measurable space $(\Omega, \mathscr{F})$ which permits the following definition. We call a family $(\theta_t)_{t\in\mathbb{R}_0^+}$ of operators $\theta_t : \Omega \to \Omega$ satisfying*

$$\forall\, s, t \in \mathbb{R}_0^+ : X_s \circ \theta_t = X_{s+t} \tag{1.1}$$

a family of **shift operators**. *Note that on an arbitrary Ω such shift operators do not necessarily exist. From now on we postulate their existence, i.e., we will only consider probability spaces on which those shift operators exist.*

Since

$$\theta_t^{-1}(X_s^{-1}(B)) = \theta_t^{-1} \circ X_s^{-1}(B) = (X_s \circ \theta_t)^{-1}(B) = X_{s+t}^{-1}(B) \in \mathscr{F}_{\geq t}^X$$

for all $s, t \in \mathbb{R}_0^+$ and $B \in \mathscr{B}(E)$, and because $\sigma(X) = \sigma(X_t^{-1}(B) : t \in \mathbb{R}_0^+, B \in \mathscr{B}(E))$, we deduce that θ_t is $\mathscr{F}_{\geq t}$–$\sigma(X)$–measurable.

In case that $\Omega = E^{\mathbb{R}_0^+}$ and $X_t = \pi_t : \Omega \to E$ (cf. Definition B.5), $t \in \mathbb{R}_0^+$, (1.1) is equivalent to

$$\forall\, \omega \in \Omega\, \forall\, s, t \in \mathbb{R}_0^+ : (\theta_t \omega)_s = \omega_{s+t}.$$

Definition 1.4 *Let $X := (X_t)_{t \in \mathbb{R}_0^+}$ be a family of E–valued random variables on some measurable space $(\Omega, \mathscr{F})$ and let $\Xi := (\xi^\nu)_{\nu \in \mathcal{P}(E)}$ be a family of probability measures on $(\Omega, \mathscr{F})$ such that $\xi^\nu_{X_0} = \nu$ for all $\nu \in \mathcal{P}(E)$ and $\xi^{(\cdot)}(A)$ is $\mathscr{B}(E)$–$\mathscr{B}([0,1])$–measurable for each $A \in \sigma(X)$, where $\xi^x := \xi^{\delta_x}$ for all $x \in E$. In addition, let $(\mathscr{G}_t)_{t \in \mathbb{R}_0^+}$ be a filtration on $(\Omega, \mathscr{F})$. Then we say that (Ξ, X) satisfies the* **Markov property** *with respect to $(\mathscr{G}_t)_{t \in \mathbb{R}_0^+}$ if*

$$\forall\, \nu \in \mathcal{P}(E)\, \forall\, Y \in \mathcal{B}(\Omega, \sigma(X))\, \forall\, t \in \mathbb{R}_0^+ : E_\nu(Y \circ \theta_t | \mathscr{G}_t) = E_{X_t}(Y) \quad \xi^\nu\text{–a.s.}, \tag{1.2}$$

where E_ν denotes the expectation with respect to ξ^ν for any $\nu \in \mathcal{P}(E)$, $E_x := E_{\delta_x}$ for every $x \in E$, and $E_{X_t}(Y) := E_{(\cdot)}(Y) \circ X_t$ for every $t \in \mathbb{R}_0^+$ and each $Y \in \mathcal{B}(\Omega, \sigma(X))$. If $\Omega = E^{\mathbb{R}_0^+}$, $X_t = \pi_t : \Omega \to E$ (cf. Definition B.5) for all $t \in \mathbb{R}_0^+$, $(\mathscr{G}_t)_{t \in \mathbb{R}_0^+} = \mathscr{F}^X$ and if (1.2) holds true, then we may simply say Ξ satisfies the Markov property.

Using the denotations of the previous definition we obtain that $E_{(\cdot)}(Y)$ is $\mathscr{B}(E)$–$\mathscr{B}(\mathbb{R})$–measurable for all $Y \in \mathcal{B}(\Omega, \sigma(X))$. Moreover, for any $t \in \mathbb{R}_0^+$ we deduce that $\xi^{X_t}(A)$ is $\mathscr{G}_t$–$\mathscr{B}([0,1])$–measurable for each $A \in \sigma(X)$, which implicates that $E_{X_t}(Y)$ is $\mathscr{G}_t$–$\mathscr{B}(\mathbb{R})$–measurable for all $Y \in \mathcal{B}(\Omega, \sigma(X))$.

Definition 1.5 *We call a semigroup $(P_t)_{t \in \mathbb{R}_0^+}$ of (sub–) Markov kernels on $(E, \mathscr{B}(E))$ a* **(sub–) Markov semigroup** *on $(E, \mathscr{B}(E))$.*

Let $(P_t)_{t \in \mathbb{R}_0^+}$ be a Markov semigroup on $(E, \mathscr{B}(E))$, let $\nu \in \mathcal{P}(E)$ and let $\pi_t : E^{\mathbb{R}_0^+} \to E$, $t \in \mathbb{R}_0^+$, be the projection as in Definition B.5. Furthermore, put $(\Omega, \mathscr{F}) := (E^{\mathbb{R}_0^+}, \mathscr{Z}(E^{\mathbb{R}_0^+}))$, where $\mathscr{Z}(E^{\mathbb{R}_0^+})$ denotes the σ–algebra generated by the family of all cylinder sets in $E^{\mathbb{R}_0^+}$. According to Theorem B.9 there exists a uniquely defined probability measure P^ν on $(\Omega, \mathscr{F})$ such that the finite dimensional distributions of the coordinate mapping process $(X_t)_{t \in \mathbb{R}_0^+}$ on $(\Omega, \mathscr{F}, P^\nu)$, defined by $X_t := \pi_t$, are given by

$$P^\nu_{(X_{t_1}, \ldots, X_{t_n})} = \nu \otimes P_{t_1} \otimes P_{t_2 - t_1} \otimes \ldots \otimes P_{t_n - t_{n-1}}$$

for each $(t_1, \ldots, t_n) \in \mathcal{H}(\mathbb{R}_0^+)$. By Theorem B.10 we have that X is a Markov process with respect to P^ν (cf. Definition B.4), which motivates the following definition:

Definition 1.6 *Let $(P_t)_{t\in\mathbb{R}_0^+}$ be a semigroup of Markov kernels on $(E,\mathscr{B}(E))$, let $\nu\in\mathcal{P}(E)$ and let $(\Omega,\mathscr{F},P^\nu)$ as well as $X:=(X_t)_{t\in\mathbb{R}_0^+}$ be as above. We call (X,P^ν) the* **canonical Markov process** *with respect to $(P_t)_{t\in\mathbb{R}_0^+}$ and with initial distribution ν. Furthermore, for any $t\in\mathbb{R}_0^+$ we call P_t* **transition kernel**. *Moreover, we denote the expectation with respect to P^ν by E_ν, and we set $P^x:=P^{\delta_x}$ as well as $E_x:=E_{\delta_x}$ for all $x\in E$. In addition, for any $t\in\mathbb{R}_0^+$ we define P^{X_t} and E_{X_t} by*

$$\forall\, A\in\mathscr{F}: P^{X_t}(A)=P^{(\cdot)}(A)\circ X_t \quad \textit{and} \quad \forall\, Y\in\mathcal{B}(\Omega,\mathscr{F}): E_{X_t}(Y)=E_{(\cdot)}(Y)\circ X_t.$$

Recall that $P^x_{X_t}$, $x\in E, t\in\mathbb{R}_0^+$, denotes the distribution of X_t under P^x, i.e., $P^x_{X_t}:=P^x\circ X_t^{-1}$. In the following let $(P_t)_{t\in\mathbb{R}_0^+}$ be a Markov semigroup on $(E,\mathscr{B}(E))$, and for any $\nu\in\mathcal{P}(E)$ let (X,P^ν) be the canonical Markov process on $(\Omega,\mathscr{F}):=(E^{\mathbb{R}_0^+},\mathscr{Z}(E^{\mathbb{R}_0^+}))$ with respect to $(P_t)_{t\in\mathbb{R}_0^+}$ and with initial distribution ν.

For any $t\in\mathbb{R}_0^+$ we define

$$\mathcal{G}:=\Big\{\{X_{t_1}\in B_1,\dots,X_{t_n}\in B_n\}:n\in\mathbb{N},0\le t_1<\dots<t_n,B_j\in\mathscr{B}(E),j=1,\dots,n\Big\}.$$

Lemma 1.7 *The mapping $P^{(\cdot)}(A):E\to\mathbb{R}$ is $\mathscr{B}(E)$–$\mathscr{B}([0,1])$–measurable for all $A\in\sigma(X)$.*

Proof Since

$$P^{(\cdot)}(A)=P^{(\cdot)}_{(X_{t_1},\dots,X_{t_n})}(B_1\times\dots\times B_n)=P_{t_1}\otimes\dots\otimes P_{t_n-t_{n-1}}(\cdot,B_1\times\dots\times B_n)$$

for all $A=(X_{t_1},\dots,X_{t_n})^{-1}(B_1\times\dots\times B_n)\in\mathcal{G}$, we deduce that $P^{(\cdot)}(A)$ is $\mathscr{B}(E)$–$\mathscr{B}([0,1])$–measurable for any $A\in\mathcal{G}$. Note that $\delta(\mathcal{G})$, the Dynkin system generated by $\mathcal{G}$, coincides with $\sigma(\mathcal{G})$, because $\mathcal{G}$ is closed under the formation of finite intersections. The aforementioned measurability extends to all $A\in\delta(\mathcal{G})=\sigma(\mathcal{G})=\sigma(X)$. □

In the following lemma we will consider a general measurable space $(\tilde\Omega,\tilde{\mathscr{F}})$, rather than $(\Omega,\mathscr{F})$. Recall that Definition 1.4 admits a general measurable space and is not restricted to the canonical model.

Lemma 1.8 *Let $\Xi:=(\xi^\nu)_{\nu\in\mathcal{P}(E)}$ be a family of probability measures on some measurable space $(\tilde\Omega,\tilde{\mathscr{F}})$ such that $\xi^\nu_{X_0}=\nu$ for all $\nu\in\mathcal{P}(E)$ and $\xi^{(\cdot)}(A)$ is $\mathscr{B}(E)$–$\mathscr{B}([0,1])$–measurable for each $A\in\sigma(X)$, where $\xi^x:=\xi^{\delta_x}$ for all $x\in E$. In addition, assume that Ξ satisfies the Markov property. Furthermore, let $(\mu_t)_{t\in\mathbb{R}_0^+}$ be a family of functions $\mu_t:E\times\mathscr{B}(E)\to[0,1]$ defined by $\mu_t(x,B)=\xi^x_{X_t}(B)$ for all $x\in E,B\in\mathscr{B}(E)$. Then $(\mu_t)_{t\in\mathbb{R}_0^+}$ is a Markov semigroup on $(E,\mathscr{B}(E))$, and*

$$\forall\,\nu\in\mathcal{P}(E):\xi^\nu_{(X_{t_1},\dots,X_{t_n})}=\nu\otimes\mu_{t_1}\otimes\mu_{t_2-t_1}\otimes\dots\otimes\mu_{t_n-t_{n-1}} \tag{1.3}$$

holds for any $(t_1,\dots,t_n)\in\mathcal{H}(\mathbb{R}_0^+)$.

Proof At first we show that μ_t, $t\in\mathbb{R}_0^+$, is a Markov kernel on $(E,\mathscr{B}(E))$. For fixed $x\in E$ we infer that $\mu_t(x,\cdot)$ is a probability measure on $(E,\mathscr{B}(E))$, since $\mu_t(x,\cdot)=\xi^x_{X_t}\in\mathcal{P}(E)$.

Moreover, for any $B \in \mathscr{B}(E)$ we have that $\mu_t(\cdot, B) = \xi^{(\cdot)}(X_t \in B)$, which is $\mathscr{B}(E)$–$\mathscr{B}([0,1])$–measurable by assumption. For all $s, t \in \mathbb{R}_0^+$ and every $B \in \mathscr{B}(E)$ let $\xi_{X_t}^{X_s}(B) := \xi_{X_t}^{(\cdot)}(B) \circ X_s$ Now observe that the Markov property of $(\xi^\nu)_{\nu \in \mathcal{P}(E)}$ yields that

$$\mu_{t-s}(X_s, B) = \xi_{X_{t-s}}^{X_s}(B) \overset{(B.7)}{=} \xi^\nu(X_t \in B | \mathscr{F}_s) \overset{(B.1)}{=} \xi^\nu(X_t \in B | X_s)$$

holds ξ^ν–a.s. for each $\nu \in \mathcal{P}(E)$, any $B \in \mathscr{B}(E)$ and all $s, t \in \mathbb{R}_0^+$ with $s \leq t$. In addition, for any $\nu \in \mathcal{P}(E)$ we denote by E_{ξ^ν} the expectation with respect to ξ^ν. Therefore,

$$\begin{aligned}
\mu_{s+t}(X_0, B) &= \xi^\nu(X_{s+t} \in B | X_0) \\
&= E_{\xi^\nu}(\xi^\nu(X_{s+t} \in B | X_0, X_s) | X_0) \\
&\overset{(B.1)}{=} E_{\xi^\nu}(\xi^\nu(X_{s+t} \in B | X_s) | X_0) \\
&\overset{(1.2)}{=} \int_{\tilde{\Omega}} \xi_{X_t}^{X_s(\omega)}(B) \xi^{(\cdot)}(\mathrm{d}\omega) \circ X_0 \\
&= \int_E \xi_{X_t}^x(B) \xi_{X_s}^{X_0}(\mathrm{d}x) \\
&= \int_E \mu_s(X_0, \mathrm{d}x) \mu_t(x, B) \\
&= \mu_s \circ \mu_t(X_0, B)
\end{aligned}$$

holds true ξ^ν–a.s. for each $\nu \in \mathcal{P}(E)$, any $B \in \mathscr{B}(E)$ and all $s, t \in \mathbb{R}_0^+$ with $s \leq t$, which shows that $\mu_{s+t} = \mu_s \circ \mu_t$, i.e., $(\mu_t)_{t \in \mathbb{R}_0^+}$ satisfies the Chapman–Kolmogorov equation. Moreover, we have

$$\forall\, x \in E \,\forall\, B \in \mathscr{B}(E) : \mu_0(x, B) = \xi_{X_0}^x(B) = \delta_x(B),$$

and thus $(\mu_t)_{t \in \mathbb{R}_0^+}$ is a Markov semigroup. In order to complete the proof it remains to show that (1.3) holds true. For this purpose let $\nu \in \mathcal{P}(E)$ and $(t_1, \ldots, t_n) \in \mathcal{H}(\mathbb{R}_0^+)$. Then we deduce that

$$\begin{aligned}
&\xi_{(X_{t_1}, \ldots, X_{t_n})}^\nu(B_0 \times B_1 \times \ldots \times B_n) \\
= &\int_{B_n} \cdots \int_{B_1} \int_{B_0} \nu(\mathrm{d}x_0) \xi_{X_{t_1}}^{x_0}(\mathrm{d}x_1) \ldots \xi_{X_{t_n - t_{n-1}}}^{x_{n-1}}(\mathrm{d}x_n) \\
= &\int_{B_n} \cdots \int_{B_1} \int_{B_0} \nu(\mathrm{d}x_0) \mu_{t_1}(x_0, \mathrm{d}x_1) \ldots \mu_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n) \\
= &\ \nu \otimes \mu_{t_1} \otimes \ldots \otimes \mu_{t_n - t_{n-1}}(B_0 \times B_1 \times \ldots \times B_n)
\end{aligned}$$

holds for all $B_0, \ldots, B_n \in \mathscr{B}(E)$, which proves the assertion. □

1.2 Feller Processes

Throughout this section let $(P_t)_{t \in \mathbb{R}_0^+}$ be a Markov semigroup on $(E, \mathscr{B}(E))$. For any $\nu \in \mathcal{P}(E)$ let (X, P^ν) be the canonical Markov process on $(\Omega, \mathscr{F}) := (E^{\mathbb{R}_0^+}, \mathscr{Z}(E^{\mathbb{R}_0^+}))$ with respect to $(P_t)_{t\in\mathbb{R}_0^+}$ and with initial distribution ν. Furthermore, let $(T_t)_{t\in\mathbb{R}_0^+}$ be the associated semigroup of transition operators, i.e., $T_t f(x) = \int_E P_t(x, \mathrm{d}y) f(y)$ for all $x \in E$ and any $t \in \mathbb{R}_0^+$.

In view of (2.11) Proposition and (2.14) Proposition in Chapter III in [RY99] we may replace the filtration $\mathscr{F}^X$ by the complete filtration $\mathscr{G}^X$ (cf. Definition A.1). In particular, we have that a family of probability measures on $(\Omega, \mathscr{F})$ which satisfies the Markov property with respect to $\mathscr{F}^X$ also satisfies the Markov property with respect to $\mathscr{G}^X$.

Let $X_\infty : \Omega \to E$ be a $\bigvee_{t\in\mathbb{R}_0^+} \mathscr{F}_t$–$\mathscr{B}(E)$–measurable function. For any $\mathscr{F}^X$–stopping time τ we define a random variable X_τ by

$$\forall\, \omega \in \Omega : X_\tau(\omega) = X_{\tau(\omega)}(\omega).$$

Definition 1.9 *For any $\mathscr{F}^X$–stopping time τ we define a* **shift operator** $\theta_\tau : \Omega \to \Omega$ *by*

$$\forall\, t \in \bar{\mathbb{R}}_0^+ : \theta_\tau = \theta_t \text{ on } \{\tau = t\},$$

where θ_∞ is given by $X_t \circ \theta_\infty = X_\infty$ for all $t \in \mathbb{R}_0^+$. Note that, confer Definition 1.3, $X_t \circ \theta_\tau = X_{t+\tau}$ for all $t \in \mathbb{R}_0^+$.

Definition 1.10 *Let $\Xi := (\xi^\nu)_{\nu\in\mathcal{P}(E)}$ be a family of probability measures on $(\Omega, \mathscr{F})$ such that $\xi^\nu_{X_0} = \nu$ for all $\nu \in \mathcal{P}(E)$ and $\xi^{(\cdot)}(A)$ is $\mathscr{B}(E)$–$\mathscr{B}([0,1])$–measurable for each $A \in \sigma(X)$, where $\xi^x := \xi^{\delta_x}$ for all $x \in E$. Then we say that Ξ satisfies the* **strong Markov property** *if*

$$E_{\xi^\nu}(Y \circ \theta_\tau | \mathscr{F}_\tau) = E_{\xi^{X_\tau}}(Y) \quad \xi^\nu\text{–a.s.} \tag{1.4}$$

holds for all $\nu \in \mathcal{P}(E)$, $Y \in \mathcal{B}(\Omega, \sigma(X))$ and $\tau \in \mathcal{S}_f(P^\nu, \mathscr{F}^X)$, where $\mathcal{S}_f(P^\nu, \mathscr{F}^X)$ denotes the set of all $\mathscr{F}^X$–stopping times which are finite P^ν–a.s.. Here, for any $\nu \in \mathcal{P}(E)$, E_{ξ^ν} denotes the expectation with respect to ξ^ν, and for each $x \in E$ we put $E_{\xi^x} := E_{\xi^{\delta_x}}$. Furthermore, for every $t \in \mathbb{R}_0^+$ and any $Y \in \mathcal{B}(\Omega, \mathscr{F})$ we adopt $E_{\xi^{X_t}}(Y) := E_{\xi^{(\cdot)}}(Y) \circ X_t$.

Lemma 1.11 *We have that (1.4) is equivalent to*

$$\forall\, x \in E\, \forall\, A \in \sigma(X)\, \forall\, \tau \in \mathcal{S}_f(P^x, \mathscr{F}^X) : P^x(\theta_\tau^{-1}(A) | \mathscr{F}_\tau) = P^{X_\tau}(A) \quad P^x\text{–a.s.}. \tag{1.5}$$

Proof "(1.4) $\Longrightarrow$ (1.5):" Let $A \in \sigma(X)$, $x \in E$ and $\tau \in \mathcal{S}_f(P^x, \mathscr{F}^X)$. Furthermore, put $Y := \mathbb{1}_A$. Then $Y \in \mathcal{B}(\Omega, \sigma(X))$ and $Y \circ \theta_\tau = \mathbb{1}_{\theta_\tau^{-1}(A)}$. With $\nu := \delta_x$ in (1.4) this shows the assertion.

"(1.5) $\Longrightarrow$ (1.4):" The proof is completely analogous to the proof of "(B.8) $\Longrightarrow$ (B.9)" in Lemma B.13, in conjunction with Lemma B.17. □

Definition 1.12 *Let $\Xi := (\xi^\nu)_{\nu\in\mathcal{P}(E)}$ be a family of probability measures on $(\Omega, \mathscr{F})$ such that $\xi^\nu_{X_0} = \nu$ for all $\nu \in \mathcal{P}(E)$ and $\xi^{(\cdot)}(A)$ is $\mathscr{B}(E)$–$\mathscr{B}([0,1])$–measurable for each $A \in \sigma(X)$, where $\xi^x := \xi^{\delta_x}$ for all $x \in E$. Then Ξ is said to satisfy the* **Feller property** *if*

(i) $\xi^{x_n}_{X_t} \to \xi^x_{X_t}$ weakly as $n \to \infty$ for any $t \in \mathbb{R}_0^+$ and all $(x_n)_{n\in\mathbb{N}} \subseteq E, x \in E$ with $x_n \to x$ as $n \to \infty$,

(ii) $X_t \stackrel{\xi^x}{\to} X_0$ *as* $t \to 0$ *for all* $x \in E$.

Lemma 1.13 $(P^\nu)_{\nu \in \mathcal{P}(E)}$ *satisfies the Feller property iff* $(T_t)_{t \in \mathbb{R}_0^+}$ *is a Feller semigroup (cf. Definition A.9), i.e.,*

(i) $\forall\, t \in \mathbb{R}_0^+ : T_t C_0(E) \subseteq C_0(E) \iff \forall\, t \in \mathbb{R}_0^+ : P_{X_t}^{x_n} \to P_{X_t}^x$ *weakly as* $n \to \infty$ *for all* $(x_n)_{n \in \mathbb{N}} \subseteq E, x \in E$ *with* $x_n \to x$ *as* $n \to \infty$,

(ii) $\forall\, f \in C_0(E)\, \forall\, x \in E : T_t f(x) \to f(x)$ *as* $t \to 0 \iff \forall\, x \in E : X_t \stackrel{P^x}{\to} X_0$ *as* $t \to 0$.

Proof (i) Let $(x_n)_{n \in \mathbb{N}} \subseteq E$ and $x \in E$ be such that $x_n \to x$. Then we have

$$
\begin{aligned}
& \forall\, t \in \mathbb{R}_0^+ : T_t C_0(E) \subseteq C_0(E) \\
\iff\ & \forall\, t \in \mathbb{R}_0^+\, \forall\, f \in C_0(E) : T_t f(x_n) \to T_t f(x) \text{ as } n \to \infty \\
\iff\ & \forall\, t \in \mathbb{R}_0^+\, \forall\, f \in C_0(E) : \int_E f(z) P_t(x_n, \mathrm{d}z) \to \int_E f(z) P_t(x, \mathrm{d}z) \text{ as } n \to \infty \\
\iff\ & \forall\, t \in \mathbb{R}_0^+\, \forall\, f \in C_0(E) : \int_E f(z) P_{X_t}^{x_n}(\mathrm{d}z) \to \int_E f(z) P_{X_t}^{x}(\mathrm{d}z) \text{ as } n \to \infty \\
\iff\ & \forall\, t \in \mathbb{R}_0^+ : P_{X_t}^{x_n} \stackrel{\mathrm{w}}{\to} P_{X_t}^x \text{ as } n \to \infty,
\end{aligned}
$$

because in our situation the concepts of weak convergence and vague convergence coincide (cf. p. 4).

(ii) Observe that

$$
\begin{aligned}
& \forall\, f \in C_0(E)\, \forall\, x \in E : T_t f(x) \to f(x) \text{ as } t \to 0 \\
\iff\ & \forall\, f \in C_0(E)\, \forall\, x \in E : \int_E P_t(x, \mathrm{d}y) f(y) \to f(x) \text{ as } t \to 0 \\
\iff\ & \forall\, f \in C_0(E)\, \forall\, x \in E : \int_E P_{X_t}^x(\mathrm{d}y) f(y) \to \int_E \delta_x(\mathrm{d}y) f(y) \text{ as } t \to 0 \\
\iff\ & \forall\, x \in E : P_{X_t}^x \stackrel{\mathrm{w}}{\to} \delta_x = P_{X_0}^x \text{ as } t \to 0 \\
\iff\ & \forall\, x \in E : X_t \stackrel{P^x}{\to} X_0 \text{ as } t \to 0,
\end{aligned}
$$

where the latter equivalence holds, since a sequence of random variables converges in probability (with respect to P^x) to some random variable which is P^x–a.s. constant iff the sequence converges in distribution (with respect to P^x) to this random variable (cf. 5.1 Theorem in Chapter I in [Bau02]).

□

Definition 1.14 *If* $(P^\nu)_{\nu \in \mathcal{P}(E)}$ *satisfies the Feller property, then we say that* (X, P^ν), $\nu \in \mathcal{P}(E)$, *is the* **canonical Feller process** *with respect to* $(P_t)_{t \in \mathbb{R}_0^+}$ *and with initial distribution* ν.

For the remainder of this section we postulate that $(P^\nu)_{\nu \in \mathcal{P}(E)}$ satisfies the Feller property.

Theorem 1.15 (Kinney's Regularity Theorem) *For any $\nu \in \mathcal{P}(E)$ the process X has an E–valued rcll P^ν–modification.*

Proof See e.g. Theorem 17.15 in [Kal01]. □

Lemma 1.16 *Let $\nu \in \mathcal{P}(E)$. Assume that X has a P^ν–modification which satisfies some property and let $\tilde{\Omega} \subseteq \Omega$ consist of all those $\omega \in \Omega$ which have this property. Then $P^{\nu^*}(\tilde{\Omega}) = 1$, where ν^* denotes the outer measure corresponding to ν.*

Proof Confer Theorem 2.4 in conjunction with Definition 2.2 in [HT94]. □

The following definition provides us with a coordinate mapping process on $\mathrm{RCLL}(\mathbb{R}_0^+, E)$.

Definition 1.17 *Put $\tilde{\Omega} := RCLL(\mathbb{R}_0^+, E) \subseteq \Omega$. Furthermore, let $\nu \in \mathcal{P}(E)$ and define $\tilde{\mathscr{F}} := \mathscr{F} \cap \tilde{\Omega} = \mathscr{B}(\tilde{\Omega})$, $\tilde{P}^\nu := P^{\nu^*}|_{\tilde{\mathscr{F}}}$ and $\tilde{X} := X|_{\tilde{\Omega}}$. Since, by Theorem 1.15 and Lemma 1.16, $P^{\nu^*}(\tilde{\Omega}) = 1$, we infer from*

$$\begin{aligned} P^\nu_{X_{t_1},\ldots,X_{t_n}} &= P^{\nu^*}\left(\left[\{(X_{t_1},\ldots,X_{t_n}) \in \cdot\} \cap \tilde{\Omega}\right] \cup \left[\{(X_{t_1},\ldots,X_{t_n}) \in \cdot\} \cap \tilde{\Omega}^{\complement}\right]\right) \\ &= P^{\nu^*}(\{(X_{t_1},\ldots,X_{t_n}) \in \cdot\} \cap \tilde{\Omega}) \qquad (1.6) \\ &= \tilde{P}^\nu_{\tilde{X}_{t_1},\ldots,\tilde{X}_{t_n}} \end{aligned}$$

for all $n \in \mathbb{N}, t_1, \ldots, t_n \in \mathbb{R}_0^+$ that we can reduce $(\Omega, \mathscr{F}, P^\nu)$ to the probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}^\nu)$ such that the finite dimensional distributions of X under P^ν and $\tilde{X}$ under $\tilde{P}^\nu$ coincide. We call $(\tilde{X}, \tilde{P}^\nu)$ the **Feller process** *with respect to $(P_t)_{t\in\mathbb{R}_0^+}$ and with initial distribution ν.*

Let us equip $\tilde{\Omega} = \mathrm{RCLL}(\mathbb{R}_0^+, E)$ with the Skorohod topology $\mathcal{T}_S$ (cf. (5.2) in Chapter 3 in [EK86]). The following lemma yields that $(\tilde{\Omega}, \mathcal{T}_S)$ is a Polish space.

Lemma 1.18 *Let $(E_1, \mathcal{T}_{E_1})$ be some Polish space. The topological space $(RCLL(\mathbb{R}_0^+, E_1), \mathcal{T}_S)$, where $\mathcal{T}_S$ denotes the Skorohod topology on $RCLL(\mathbb{R}_0^+, E_1)$, is a Polish space.*

Proof See 5.6 Theorem in Chapter 3 in [EK86]. □

Theorem 1.19 *$(\tilde{P}^\nu)_{\nu\in\mathcal{P}(E)}$ satisfies the strong Markov property.*

Proof See Theorem 1 in Section 3 of Chapter 2 in [Chu82] or Theorem 17.17 in [Kal01]. □

1.3 Feller Diffusions

In this section we will accomplish the main purpose of this chapter, for that we will give a first definition of diffusion processes. Since the diffusion processes which we are going to define in this section evolve from Feller processes, we will refer to them as Feller diffusions.

Our proceeding during this section is mainly based upon Chapter 17 in [Kal01], §6 of Chapter Five and §1 of Chapter Ten in [Dyn65I] as well as §5 of Chapter Thirteen in [Dyn65II]. The main ideas regarding the proofs are due to [Kal01].

For each $d \in \mathbb{N}$ we denote the set of all symmetric real positive semidefinite $d \times d$ matrices by $\mathcal{M}_d$. In addition, for each domain $U \subseteq \mathbb{R}^d$ and for any $a : U \to \mathcal{M}_d$ and $x \in U$ we denote the entries $a(x)_{ij}$; $i, j = 1, \ldots, d$; of $a(x)$ by $a_{ij}(x)$. Moreover, for any function $b : U \to \mathbb{R}^d$ we denote by b_i the coordinate mapping.

Let $(P_t)_{t\in\mathbb{R}_0^+}$ be a Markov semigroup on $(E, \mathscr{B}(E))$, and assume that the corresponding semigroup $(T_t)_{t\in\mathbb{R}_0^+}$ of transition operators is a Feller semigroup. For any $\nu \in \mathcal{P}(E)$ let (X, P^ν) be the Feller process on $(\Omega, \mathscr{F}) := (\text{RCLL}(\mathbb{R}_0^+, E), \mathscr{B}[\text{RCLL}(\mathbb{R}_0^+, E)])$ with respect to $(P_t)_{t\in\mathbb{R}_0^+}$ and with initial distribution ν. Furthermore, let T be the generator of $(T_t)_{t\in\mathbb{R}_0^+}$ with domain $\mathcal{D} \subseteq C_0(\mathbb{R}^d)$.

1.3.1 Feller Diffusions in $\mathbb{R}^d$

This subsection is devoted to the development of the notion of Feller diffusions in $\mathbb{R}^d$. In the next subsection we will utilise this in order to introduce the concept of Feller diffusions in a domain $U \subseteq \mathbb{R}^d$.

In the following we will consider stochastic processes $M^f := (M_t^f)_{t\in\mathbb{R}_0^+}$, $f \in \mathcal{D}$, defined by

$$\forall\, t \in \mathbb{R}_0^+ : M_t^f := f \circ X_t - f \circ X_0 - \int_{[0,t]} (Tf) \circ X_s \, \lambda(\mathrm{d}s). \tag{1.7}$$

Lemma 1.20 *Let $f \in \mathcal{D}$. For any $\nu \in \mathcal{P}(E)$ the process (M^f, P^ν) is an $\mathscr{F}^X$–martingale. Moreover, in particular this implies that*

$$\forall\, x \in E : E_x(f \circ X_\tau) = f(x) + E_x \int_{[0,\tau]} Tf \circ X_s \, \lambda(ds) \tag{1.8}$$

holds for every bounded $\mathscr{F}^X$–stopping stopping time τ. We call (1.8) **Dynkin's formula**.

Proof Fix an arbitrary $\nu \in \mathcal{P}(E)$. At first observe that

$$\begin{aligned} M_{t+h}^f - M_t^f &= f \circ X_{t+h} - f \circ X_t - \int_{(t,t+h]} Tf \circ X_s \, \lambda(\mathrm{d}s) \\ &= f \circ X_h \circ \theta_t - f \circ X_0 \circ \theta_t - \int_{(0,h]} Tf \circ X_s \circ \theta_t \, \lambda(\mathrm{d}s) \\ &= M_h^f \circ \theta_t. \end{aligned}$$

holds for all $t, h \in \mathbb{R}_0^+$. Moreover, $M_h^f \in \mathcal{B}(\Omega, \sigma(X))$, and thus we deduce from the Markov property (cf. Lemma B.13) that

$$\forall\, t, h \in \mathbb{R}_0^+ : E_\nu(M_{t+h}^f - M_t^f | \mathscr{F}_t) = E_\nu(M_h^f \circ \theta_t | \mathscr{F}_t) = E_{X_t}(M_h^f) \overset{(*)}{=} 0 \quad P^\nu\text{–a.s.},$$

where $(*)$ holds, since

$$E_{X_t}(M_h^f) = \left(T_h f - f - \int_{[0,h]} T_s T f \, \lambda(\mathrm{d}s) \right) \circ X_t = 0.$$

holds true by Theorem B.22 (i). Therefore, because M^f is $\mathscr{F}^X$–adapted, we infer that (M^f, P^ν) is an $\mathscr{F}^X$–martingale. Now consider a bounded $\mathscr{F}^X$–stopping time τ. By the Optional Sampling Theorem (cf. Theorem B.18) we have $E_x(M^f_\tau) = E_x(M^f_0) = 0$, $x \in E$, and hence, by taking the expectation in (1.7), we obtain

$$\forall\, x \in E : E_x(f \circ X_\tau) - f(x) - E_x \int_{[0,\tau]} Tf \circ X_s \, \lambda(\mathrm{d}s) = E_x(M^f_\tau) = 0,$$

which shows that (1.8) holds. □

For any $h \in \mathbb{R}^+_0$ we define an $\mathscr{F}^X$–stopping time τ_h by

$$\tau_h := \inf\{t \in \mathbb{R}^+_0 : \rho_E \circ (X_0, X_t) > h\}.$$

Note that τ_h is indeed an $\mathscr{F}^X$–stopping time, since $\mathscr{F}^X$ is right–continuous, and moreover observe that $x \in E$ is absorbing with respect to P^x iff $\tau_0 \equiv \infty$ P^x–a.s..

Lemma 1.21 *For any nonabsorbing $x \in E$ there exists an $h_0 \in \mathbb{R}^+_0$ such that $E_x(\tau_{h_0}) < \infty$ for all $h \in [0, h_0]$.*

Proof Let $x \in E$ be nonabsorbing. Then, since x is nonabsorbing, there exist $t_0, \varepsilon_0 > 0$ and $p_0 \in [0, 1)$ such that $P_{t_0}(x, B[x, \varepsilon_0]) \le p_0$. Consider an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in E converging to x. We deduce from Lemma 1.13 (i) that $P^{x_n} \to P^x$ weakly as $n \to \infty$. By the Portmanteau Theorem (cf. Theorem 3.25 in [Kal01]) this implies that

$$\limsup_{n\to\infty} P^{x_n}_{X_{t_0}}(B[x, \varepsilon_0]) \le P^x_{X_{t_0}}(B[x, \varepsilon_0]),$$

i.e., $P^{(\cdot)}_{X_{t_0}}(B[x, \varepsilon_0])$ is upper semicontinuous in x. Therefore, for each $\delta > 0$ there exists an $h_0 > 0$ such that

$$P^y_{X_{t_0}}(B[x, \varepsilon_0]) \le P^x_{X_{t_0}}(B[x, \varepsilon_0]) + \delta \le p_0 + \delta, \tag{1.9}$$

holds for all $y \in B[x, h_0]$. This can be seen as follows: By definition of the limit superior there exist $k, n_1, \dots, n_k, \in \mathbb{N}$ such that

$$\forall \delta > 0\, \forall\, n \in \mathbb{N} \setminus \{n_1, \dots, n_k\} : P^{x_n}_{X_{t_0}}(B[x, \varepsilon_0]) \le P^x_{X_{t_0}}(B[x, \varepsilon_0]) + \delta. \tag{1.10}$$

Now assume there were a $\delta > 0$ such that for every $h > 0$ there exists some $y_h \in B[x, h]$ with $P^{y_h}_{X_{t_0}}(B[x, \varepsilon_0]) > P^x_{X_{t_0}}(B[x, \varepsilon_0]) + \delta$. Then we could construct a sequence $(z_n)_{n\in\mathbb{N}}$ by $z_n := y_{1/n}$ for all $n \in \mathbb{N}$. Note that $z_n \to x$ as $n \to \infty$ and $P^{z_n}_{X_{t_0}}(B[x, \varepsilon_0]) > P^x_{X_{t_0}}(B[x, \varepsilon_0]) + \delta$ for all $n \in \mathbb{N}$. But this would pose a contradiction to (1.10).

Choose some $\delta > 0$ with $p_\delta := p_0 + \delta < 1$, and let $h_0 \in (0, \varepsilon_0]$ be such that (1.9) holds for all $y \in B[x, h_0]$. Then we obtain that

$$\begin{aligned} P^x(\tau_{h_0} > nt_0) &\le P^x_{(X_{t_0}, X_{2t_0}, \dots, X_{nt_0})}(B[x, h_0]^n) \\ &= \int_{B[x,h_0]} \int_{B[x,h_0]} \cdots \int_{B[x,h_0]} P^x_{X_{t_0}}(\mathrm{d}x_1) P^{x_1}_{X_{t_0}}(\mathrm{d}x_2) \dots P^{x_{n-1}}_{X_{t_0}}(\mathrm{d}x_n) \end{aligned}$$

$$\leq \; p_\delta^n$$

holds for all $n \in \mathbb{N}$. It follows that

$$E_x(\tau_{h_0}) = \int_{[0,\infty)} P^x(\tau_{h_0} \geq s)\lambda(\mathrm{d}s) \leq t_0 \sum_{n\in\mathbb{N}_0} P^x(\tau_{h_0} \geq nt_0) \leq t_0 \sum_{n\in\mathbb{N}_0} p_\delta^n = \frac{t_0}{1-p_\delta} < \infty,$$

since $|p_\delta| < 1$. Of course, this property extends to all $h \in [0, h_0]$. □

Definition 1.22 *We denote by $\mathcal{D}_D$ the set of all $f \in \mathcal{B}(E)$ such that*

$$\lim_{h\downarrow 0} \frac{E_x(f \circ X_{\tau_h}) - f(x)}{E_x(\tau_h)}$$

exists and is finite for all nonabsorbing $x \in E$. Furthermore, the linear operator L on $\mathcal{D}_D$, defined by $Lf(x) = 0$, $f \in \mathcal{D}_D$, for any absorbing $x \in E$ and

$$\forall\, f \in \mathcal{D}_D : Lf(x) = \lim_{h\downarrow 0} \frac{E_x(f \circ X_{\tau_h}) - f(x)}{E_x(\tau_h)} \tag{1.11}$$

for each nonabsorbing $x \in E$, is referred to as **Dynkin's characteristic operator***.*

Theorem 1.23 *We have that $\mathcal{D} \subseteq \mathcal{D}_D$, and T is the restriction of Dynkin's characteristic operator onto $\mathcal{D}$.*

Proof Let h_0 be as in Lemma 1.21 and fix some $f \in \mathcal{D}$. Initially, let $x \in E$ be absorbing. Then we have for any $t \in \mathbb{R}_0^+$ that $T_t f(x) = \int_E P_t(x, \mathrm{d}y) f(y) = f(x)$. Thus, by Theorem B.22 (ii), $Tf(x) = T_t T f(x) = \frac{\mathrm{d}}{\mathrm{d}t}(T_t f(x)) = 0$, $t \in \mathbb{R}_0^+$. Now let $x \in E$ be nonabsorbing. For all $h \in [0, h_0]$ and $t \in \mathbb{R}_0^+$ we have that $\tau_h \wedge t$ is a $\mathscr{F}^X$–stopping time which is bounded above by t, and thus we infer from Dynkin's formula (cf. Lemma 1.20) that

$$\forall\, t \in \mathbb{R}_0^+ \,\forall\, h \in [0, h_0] : E_x(f \circ X_{\tau_h \wedge t}) - f(x) = E_x \int_{[0,\tau_h\wedge t]} Tf \circ X_s \,\lambda(\mathrm{d}s). \tag{1.12}$$

Since $\tau_h \wedge t$ is bounded above by τ_h and, by Lemma 1.21, $E_x(\tau_h) < \infty$ for all $h \in [0, h_0]$, we can apply the Dominated Convergence Theorem in (1.12) at taking the limit as $t \to \infty$ and obtain that

$$\begin{aligned}
\frac{E_x(f \circ X_{\tau_h}) - f(x)}{E_x(\tau_h)} &= \frac{E_x\left(\int_{[0,\tau_h]} Tf \circ X_s \,\lambda(\mathrm{d}s)\right)}{E_x(\tau_h)} \\
&= \frac{E_x\left(\int_{[0,\tau_h)} (Tf \circ X_s - Tf \circ X_0 + Tf \circ X_0)\,\lambda(\mathrm{d}s)\right)}{E_x(\tau_h)} \\
&= \frac{E_x\left(\int_{[0,\tau_h)} (Tf \circ X_s - Tf \circ X_0)\,\lambda(\mathrm{d}s)\right)}{E_x(\tau_h)} + Tf(x)
\end{aligned} \tag{1.13}$$

holds for any $h \in [0, h_0]$. Note that because of the right-continuity of X, we have that $E_x(\tau_h) > 0$ for all $h > 0$. Fix some arbitrary $\varepsilon > 0$, then Tf being continuous yields that there exists a $\delta_\varepsilon > 0$ such that $|Tf(y) - Tf(x)| \leq \varepsilon$ for all $y \in B[x, \delta_\varepsilon]$. Now choose an arbitrary $h_\varepsilon \in [0, h_0 \wedge \delta_\varepsilon]$. Then we have for any $\omega \in \Omega$ that $\rho_E(X_0(\omega), X_s(\omega)) \leq \delta_\varepsilon$ for all $s \in [0, \tau_{h_\varepsilon}(\omega))$. This results in

$$\left| \frac{E_x\left(\int_{[0,\tau_{h_\varepsilon})} (Tf \circ X_s - Tf \circ X_0)\, \lambda(\mathrm{d}s)\right)}{E_x(\tau_{h_\varepsilon})} \right| \leq \frac{E_x\left(\int_{[0,\tau_{h_\varepsilon})} |Tf \circ X_s - Tf \circ X_0|\, \lambda(\mathrm{d}s)\right)}{E_x(\tau_{h_\varepsilon})} \leq \frac{\varepsilon E_x(\tau_{h_\varepsilon})}{E_x(\tau_{h_\varepsilon})} = \varepsilon,$$

which shows that $\lim_{h \downarrow 0} \frac{E_x\left(\int_{[0,\tau_h)} (Tf \circ X_s - Tf \circ X_0)\, \lambda(\mathrm{d}s)\right)}{E_x(\tau_h)} = 0$. Therefore, by means of (1.13), we infer that

$$\lim_{h \downarrow 0} \frac{E_x(f \circ X_{\tau_h}) - f(x)}{E_x(\tau_h)} = \lim_{h \downarrow 0} \frac{E_x\left(\int_{[0,\tau_h)} (Tf \circ X_s - Tf \circ X_0)\, \lambda(\mathrm{d}s)\right)}{E_x(\tau_h)} + Tf(x) = Tf(x).$$

□

For the remaining part of this chapter we fix an arbitrary $d \in \mathbb{N}$ and consider the locally compact Polish space $\mathbb{R}^d$ equipped with the Euclidean norm, which we denote by $\|\cdot\|_2$. In addition, we denotate by the same symbol, $\|\cdot\|_2$, the function $\|\cdot\|_2 : \mathscr{L}^0(\Omega) \to \mathbb{R}_0^+$ defined by $\|Y\|_2(\omega) = \|Y(\omega)\|_2$ for any $Y \in \mathscr{L}^0(\Omega)$ and all $\omega \in \Omega$. This ambiguity of the symbol $\|\cdot\|_2$ is somewhat intuitive and should not lead to any misinterpretations.

Definition 1.24 *A $\mathcal{B}(\mathbb{R}^d)$–valued linear operator L with domain $\mathcal{D}_L$, $C_K^2(\mathbb{R}^d) \subseteq \mathcal{D}_L \subseteq \mathcal{B}(\mathbb{R}^d)$, is called* **local** *if for every $f \in C_K^2(\mathbb{R}^d)$ the property $Lf(x) = 0$ holds for any $x \in \mathbb{R}^d$ with $f \equiv 0$ on $B[x, \varepsilon]$ for some $\varepsilon > 0$.*

Definition 1.25 *Let L be a $\mathcal{B}(\mathbb{R}^d)$–valued linear operator with domain $\mathcal{D}_L$, $C_K^2(\mathbb{R}^d) \subseteq \mathcal{D}_L \subseteq \mathcal{B}(\mathbb{R}^d)$. We say that L satisfies the* **local positive maximum principle** *if*

$$\left[f \in C_K^2(\mathbb{R}^d), x \in \mathbb{R}^d, \varepsilon > 0 : f^+(y) \leq f(x)\, \forall\, y \in B[x, \varepsilon]\right] \Longrightarrow [Lf(x) \leq 0].$$

Let $C_1, C_2 \subseteq \mathbb{R}^d$ be disjoint closed sets. In the proof of the following lemma, as well as on several occasions throughout this thesis, we will utilise a so–called **cut–off function** $\varphi \in C_K^\infty(\mathbb{R}^d, [0,1])$ with $\varphi \equiv 1$ on C_1 and $\varphi \equiv 0$ on C_2. For the proof of existence as well as a construction of such a cut–off function see (7.2) and (7.4) Remark in [BJ73].

Lemma 1.26 *Let L be a $\mathcal{B}(\mathbb{R}^d)$–valued linear operator with domain $\mathcal{D}_L$, $C_K^2(\mathbb{R}^d) \subseteq \mathcal{D}_L \subseteq \mathcal{B}(\mathbb{R}^d)$, which is local and satisfies the positive maximum principle (cf. Definition A.11). Then L satisfies the local positive maximum principle.*

Proof Let $f \in C_K^2(\mathbb{R}^d)$ have a nonnegative local maximum at $x_0 \in \mathbb{R}^d$. Then there exists an $\varepsilon > 0$ with $f(x_0) \geq f^+(x)$ for all $x \in B[x_0, \varepsilon]$. Moreover, there exists a cut–off function $\varphi \in C_K^2(\mathbb{R}^d)$ with $\varphi \equiv 1$ on $B[x_0, \varepsilon/2]$, $\varphi \equiv 0$ on $\mathbb{R}^d \setminus B[x_0, \varepsilon]$ and $\varphi(x) \in [0,1]$ for all $x \in B[x_0, \varepsilon] \setminus B[x_0, \varepsilon/2]$. Furthermore, choose some $\psi \in C_K^2(\mathbb{R}^d)$ with $\psi = \mathbf{1} - \varphi$ on $\operatorname{supp}(f)$.

Note that the product of functions in $C_K^2(\mathbb{R}^d)$ is again an element of $C_K^2(\mathbb{R}^d)$. It follows from the positive maximum principle that $L(\varphi \cdot f)(x_0) \leq 0$, because $\varphi \cdot f$ has a nonnegative global maximum at x_0. Furthermore, L being local implies that $L(\psi \cdot f)(x_0) = 0$, since $\psi \cdot f \equiv 0$ on $B[x_0, \varepsilon/2]$. By the linearity of L we obtain

$$Lf(x_0) = L(\varphi \cdot f)(x_0) + L(\psi \cdot f)(x_0) \leq 0.$$

□

Theorem 1.27 *If $C_K^2(\mathbb{R}^d) \subseteq \mathcal{D}$, then T is local iff for every $\nu \in \mathcal{P}(\mathbb{R}^d)$ the process X is continuous P^ν–a.s., i.e., P^ν–a.a. trajectories of X are continuous on $\mathbb{R}_0^+$. Moreover, if either of these two equivalent properties holds, then there exist $(a_{ij})_{i,j=1,\dots,d} \in C(\mathbb{R}^d, \mathcal{M}_d)$, $(b_i)_{i=1,\dots,d} \in C(\mathbb{R}^d, \mathbb{R}^d)$ and $c \in C(\mathbb{R}^d, \mathbb{R}_0^+)$ such that*

$$Tf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} f(x) - cf(x). \tag{1.14}$$

for all $f \in C_K^2(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$. The coefficients $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ are called **diffusion coefficient** *and* **drift coefficient**, *respectively.*

Proof Firstly, we presume that X is P^ν–a.s. continuous for any probability measure ν. Let $f \in C_K^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $\varepsilon > 0$ such that $f \equiv 0$ on $B[x, \varepsilon]$. For the time being, assume that x is nonabsorbing. Note that because of the right-continuity of X we have that $E_x(\tau_h) > 0$ for all $h > 0$, and because of the continuity of X we have that $\|X_0 - X_{\tau_\varepsilon}\|_2 = \varepsilon$ P^x–a.s.. Therefore, $E_x(f \circ X_{\tau_h}) = 0$ for all $h \in [0, \varepsilon]$, and thus, since $f(x) = 0$, we infer that

$$Tf(x) \overset{T.1.23}{=} \lim_{h \downarrow 0} \frac{E_x(f \circ X_{\tau_h}) - f(x)}{E_x(\tau_h)} = 0.$$

If, however, x is absorbing, then Theorem 1.23 yields that $Tf(x) = 0$.

Conversely, we presume T to be local, and we choose an arbitrary $x \in \mathbb{R}^d$ and $k, m \in \mathbb{N}$ with $1/k < m$. Furthermore, let $f \in C_K^2(\mathbb{R}^d)$ be nonnegative with $\operatorname{supp}(f) = \{y \in \mathbb{R}^d : \|y - x\|_2 \in [1/k, m]\}$. Then $Tf \equiv 0$ on $B(x, 1/k)$, because T is local and for each $y \in B(x, 1/k)$, there is a neighbourhood $N \subseteq B(x, 1/k)$ of y. Now Tf being continuous yields that even $Tf \equiv 0$ on $B[x, 1/k]$. Furthermore, $X_t(\omega) \in B[X_0, 1/k]$ for all $t \in [0, \tau_{1/k}), \omega \in \Omega$. Hence we infer from Lemma 1.21 and the Optional Sampling Theorem (cf. Theorem B.18) that, under P^x, $(f \circ X_{t \wedge \tau_{1/k}})_{t \in \mathbb{R}_0^+} \underset{a.s.}{\overset{P^x-}{=}} (M^f_{t \wedge \tau_k})_{t \in \mathbb{R}_0^+}$ is a martingale with respect to $\mathscr{F}^X$. Moreover, $E_x(f \circ X_0) = 0$, and thus, by the martingale property mentioned above, we infer that $E_x(f \circ X_{t \wedge \tau_{1/k}}) = 0$ for all $t \in \mathbb{R}_0^+$. Taking the limit as $t \to \infty$ we deduce from the Dominated Convergence Theorem that $E_x(f \circ X_{\tau_{1/k}}) = 0$. Because f is nonnegative, it follows that $f \circ X_{\tau_{1/k}} = 0$ P^x–a.s., i.e., $\|X_{\tau_{1/k}} - x\|_2 \in [0, 1/k] \cup [m, \infty]$ P^x–a.s.. Letting $m \to \infty$, we infer that

$$\forall\, k \in \mathbb{N} : P^x(\|X_{\tau_{1/k}} - x\|_2 \leq 1/k) = 1. \tag{1.15}$$

Let $\nu \in \mathcal{P}(\mathbb{R}^d)$. Since $x \in \mathbb{R}^d$ was chosen arbitrarily, we obtain

$$\forall\, k \in \mathbb{N} : P^\nu\left(\bigcap_{t \in \mathbb{N}_0} \theta_t^{-1}\{\|X_{\tau_{1/k}} - X_0\|_2 \leq 1/k\}\right) = 1. \tag{1.16}$$

To see this, put $A := \{\|X_{\tau_{1/k}} - x\|_2 \leq 1/k\}$ and observe that $P^{X_t}(A) \equiv 1$ holds true, because (1.15) holds for every $x \in \mathbb{R}^d$. Hence the Markov property of $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ yields that $P^{X_t}(A) = P^x(\theta_t^{-1}(A)|\mathscr{F}_t)$, which results in $P^x(\theta_t^{-1}(A)) = 1$ for all $x \in \mathbb{R}^d$. Then (1.16) follows from Lemma B.17 and the fact that the countable union of P^ν–null sets is again a P^ν–null set.

Observe that

$$\theta_{t-1/n}^{-1}\{\|X_{\tau_{1/k}} - X_0\|_2 \leq 1/k\} = \{\|X_{t+\tau_{1/k}-1/n} - X_{t-1/n}\|_2 \leq 1/k\}$$

holds for any $t \in \mathbb{R}_0^+$ and all $k, n \in \mathbb{N}$. Therefore, we deduce from (1.16) that

$$\forall\, t \in \mathbb{R}_0^+ \,\forall\, k, n \in \mathbb{N}, 1/n \leq \tau_{1/k} : P^\nu\left(\|X_{t+\tau_{1/k}-1/n} - X_{t-1/n}\|_2 \leq 1/k\right) = 1.$$

Letting $k \to \infty$ and $n \to \infty$ such that $\tau_{1/k} - 1/n \downarrow 0$ results in $\|\Delta X_t\|_2 = 0$ P^ν–a.s. for all $t \in \mathbb{R}_0^+$, where $\Delta X_t := X_{t+0} - X_{t-0}$. The right-continuity of X yields that $\|\Delta X_t\|_2 = 0$ P^ν–a.s. for all $t \in \mathbb{R}_0^+$, which shows that X is continuous P^ν–a.s..

It remains to show that (1.14) holds. To this end we fix some arbitrary $x \in \mathbb{R}^d$ as well as $\varepsilon > 0$, and we consider $f_{ij}^x, f_i^x \in C_K^2(\mathbb{R}^d)$, $i, j = 1, \ldots, d$, defined on $B[x, \varepsilon]$ by

$$\forall\, y \in B[x,\varepsilon] : \quad f_{ij}^x(y) = (y_i - x_i)(y_j - x_j), \quad f_i^x(y) = y_i - x_i.$$

Furthermore, choose $f_{ij}, f_i, f_0 \in C_K^2(\mathbb{R}^d)$, $i, j = 1, \ldots, d$, such that

$$f_{ij} = y_i y_j, \quad f_i(y) = y_i, \quad f_0(y) = 1$$

holds for all $y \in B[x, \varepsilon]$, and moreover put

$$a_{ij}^x := Tf_{ij}^x(x), \quad b_i^x := Tf_i^x(x), \quad c^x := -Tf_0(x). \tag{1.17}$$

Observe that a_{ij}^x, b_i^x and c^x are well defined, i.e., they do not depend on the actual choice of $f_{ij}^x, f_i^x, f_0 \in C_K^2(\mathbb{R}^d)$, $i, j = 1, \ldots, d$, with $f_{ij}^x(y) = (y_i - x_i)(y_j - x_j)$, $f_i^x(y) = y_i - x_i$ and $f_0(y) = 1$ for all $y \in B[x, \varepsilon]$. To see this, choose $f_{ij}^x, g_{ij}^x, f_i^x, g_i^x, f_0, g_0 \in C_K^2(\mathbb{R}^d)$ with the properties mentioned above. Then T being local results in

$$T(f_{ij}^x - g_{ij}^x)(x) = T(f_i^x - g_i^x)(x) = T(f_0 - g_0)(x) = 0,$$

since $f_{ij}^x - g_{ij}^x = f_i^x - g_i^x = f_0 - g_0 = 0$ on $B[x, \varepsilon]$. Now we deduce from the linearity of T that a_{ij}^x, b_i^x and c^x are well defined. It follows from the linearity of T that

$$\begin{aligned} Tf_{ij}(x) &= T(f_{ij}^x + x_j f_i + x_i f_j - x_i x_j f_0)(x) = a_{ij}^x + x_j b_i^x + x_i b_j^x - x_i x_j c^x, \\ Tf_i(x) &= T(f_i^x + x_i f_0)(x) = b_i^x - x_i c^x. \end{aligned} \tag{1.18}$$

Let $\varphi \in C_K^2(\mathbb{R}^d)$ be such that $\varphi|_{B[x,\varepsilon]}$ is a second–degree polynomial (cf. Definition A.2). Then there exist $\alpha_{ij}, \beta_i, \gamma \in \mathbb{R}$, $i, j = 1, \ldots, d$, such that

$$\varphi(x) = \sum_{i,j=1}^d \alpha_{ij} x_i x_j + \sum_{i=1}^d \beta_i x_i + \gamma.$$

We infer that

$$\frac{\partial}{\partial x_i}\varphi(x) = \sum_{j=1}^{d}(\alpha_{ij}+\alpha_{ji})x_j + \beta_i \quad \text{and} \quad \frac{\partial^2}{\partial x_i \partial x_j}\varphi(x) = \alpha_{ij}+\alpha_{ji},$$

and obtain

$$\begin{aligned} T\varphi(x) &= \sum_{i,j=1}^{d}\alpha_{ij}Tf_{ij}(x) + \sum_{i=1}^{d}\beta_i Tf_i(x) + \gamma Tf_0(x) \\ &= \sum_{i,j=1}^{d}\alpha_{ij}(a_{ij}^x + x_j b_i^x + x_i b_j^x - x_i x_j c^x) + \sum_{i=1}^{d}\beta_i(b_i^x - x_i c^x) - \gamma c^x \\ &= \frac{1}{2}\sum_{i,j=1}^{d}a_{ij}^x(\alpha_{ij}+\alpha_{ji}) + \sum_{i=1}^{d}b_i^x\left(\sum_{j=1}^{d}(\alpha_{ij}+\alpha_{ji})x_j + \beta_i\right) \\ &\quad -c^x\left(\sum_{i,j=1}^{d}\alpha_{ij}x_i x_j + \sum_{i=1}^{d}\beta_i x_i + \gamma\right) \\ &= \frac{1}{2}\sum_{i,j=1}^{d}a_{ij}^x\frac{\partial^2}{\partial x_i\partial x_j}\varphi(x) + \sum_{i,j=1}^{d}b_i^x\frac{\partial}{\partial x_i}\varphi(x) - c^x\varphi(x). \end{aligned}$$

By Lemma 1.26 in conjunction with Theorem B.23, T satisfies the local positive maximum principle, and thus we infer from (1.17) that $c^x = -Tf_0(x) \geq 0$, because f_0 has a nonnegative local maximum at x. Moreover, $(a_{ij}^x)_{i,j=1,\dots,d}$ is positive semi–definite. This can be seen by applying the local positive maximum principle to the functions $g_\xi := -\left(\sum_{i=1}^{d}\xi_i f_i^x\right)^2$, $\xi \in \mathbb{R}^d$, which have a nonnegative local maximum at x. This implies that $\xi^T(a_{ij}^x)_{i,j=1,\dots,d}\xi = -Tf_\xi(x) \geq 0$ for all $\xi \in \mathbb{R}^d$.

So far the choice of $x \in \mathbb{R}^d$ was arbitrary, i.e., the above constructions can be made on the lines of our considerations above for all $x \in \mathbb{R}^d$. Having done this, we can define $a_{ij}, b_i, c : \mathbb{R}^d \to \mathbb{R}$, $i,j = 1,\dots,d$, by

$$\forall\, x \in \mathbb{R}^d : a_{ij}(x) = a_{ij}^x, \quad b_i(x) = b_i^x, \quad c(x) = c^x.$$

Note that (1.18) implies that $a_{ij}, b_i, c \in C(\mathbb{R}^d)$ and $a_{ij} = a_{ji}$ for all $i,j = 1,\dots,d$. Therefore, (1.14) holds at x for any function which coincides with a second–degree polynomial on $B[x,\varepsilon]$.

Again choose arbitrary $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Furthermore, let $f \in C_K^2(\mathbb{R}^d)$ and let $g \in C_K^2(\mathbb{R}^d)$ be such that on $B[x,\varepsilon]$ g coincides with a second–order Taylor expansion (cf. Definition A.2) of f around x, i.e.,

$$\begin{aligned} g(y) &= f(x) + \sum_{i=1}^{d}\frac{\partial}{\partial x_i}f(x)(y-x) \\ &\quad +2\sum_{i=1}^{d}\sum_{j>i}\frac{\partial^2}{\partial x_i\partial x_j}f(x)(y_i-x_i)(y_j-x_j) + \frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}f(x)(y_i-x_i)^2. \end{aligned}$$

holds for all $y \in B[x, \varepsilon]$. Hence, on $B[x, \varepsilon]$ the function g coincides with a second–degree polynomial, and thus we obtain by our considerations above that (1.14) holds at x for g. In order to extend this result to f we will utilise a result about the remainder term in the Taylor formula and the fact that $g(x) = f(x)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ converging to x. By the qualitative Taylor formula, which can be found in Section 4 of Chapter 2 in [Kön97], we have that $([f(x_n) - g(x_n)]\|x_n - x\|_2^{-2}) \to 0$ as $n \to \infty$. Therefore, there exists an $\varepsilon > 0$ such that any $\psi_\varepsilon \in C_K^2(\mathbb{R}^d)$ given on $B[x, \varepsilon]$ by

$$\forall\, y \in B[x, \varepsilon] : \psi_\varepsilon(y) = f(y) - g(y) - \varepsilon\|x - y\|_2^2$$

has a nonnegative local maximum at x, since $\psi_\varepsilon(x) = 0$, which, by the local positive maximum principle, yields that

$$\forall \varepsilon > 0 : T\psi_\varepsilon(x) = Tf(x) - Tg(x) - \varepsilon \sum_{i=1}^{d} a_{ii}(x) \le 0.$$

Since the inequality holds for all $\varepsilon > 0$, we infer that

$$Tf(x) - Tg(x) = \lim_{\varepsilon\to 0} T\psi_\varepsilon(x) \le 0,$$

Analogously we obtain that $Tf(x) - Tg(x) \le 0$, which shows that $Tf(x) = Tg(x)$, and consequently (1.14) holds at x for f. From that it follows the assertion, because $x \in \mathbb{R}^d$ was chosen arbitrarily. □

Presume that $C_K^2(\mathbb{R}^d) \subseteq \mathcal{D}$, and furthermore assume that T is local. In addition, define $\acute{\Omega} := C(\mathbb{R}_0^+, \mathbb{R}^d) \subseteq \Omega$, $\acute{\mathscr{F}} := \mathscr{F} \cap \acute{\Omega}$, $\acute{X} := X|_{\acute{\Omega}}$ and $\acute{P}^\nu := P^\nu|_{\acute{\mathscr{F}}}$ for any $\nu \in \mathcal{P}(\mathbb{R}^d)$. Since, by means of Theorem 1.27, $P^\nu(\acute{\Omega}) = 1$, we infer from an argument as in (1.6) that we can reduce $(\Omega, \mathscr{F}, P^\nu)$ to the probability space $(\acute{\Omega}, \acute{\mathscr{F}}, \acute{P}^\nu)$ such that the finite dimensional distributions of X under P^ν and $\acute{X}$ under $\acute{P}^\nu$ coincide. For every $\nu \in \mathcal{P}(\mathbb{R}^d)$ we call $(\acute{X}, \acute{P}^\nu)$ the **Feller diffusion** in $\mathbb{R}^d$ with respect to T and with initial distribution ν. Let $\acute{\mathscr{F}}_t := \sigma(\acute{X}_s : s \in [0, t])$ for all $t \in \mathbb{R}_0^+$, which gives a filtration $\mathscr{F}^{\acute{X}} := (\acute{\mathscr{F}}_t)_{t\in\mathbb{R}_0^+}$ on $(\acute{\Omega}, \acute{\mathscr{F}})$.

Above we have constructed Feller diffusions via Markov semigroups. In the next Theorem one starts with a certain operator L, which is the generator of a Feller semigroup, and one obtains a corresponding Feller diffusion as well as a Markov semigroup of transition kernels.

Theorem 1.28 *Let L be the generator of a Feller semigroup of transition operators, and presume that L satisfies (1.14) for some coefficients $a_{ij}, b_i, c \in C_b^{0,\theta}(\mathbb{R}^d)$ for some $\theta > 0$ and all $i, j \in \{1, \ldots, d\}$. If $(a_{ij})_{i,j=1,\ldots,d}$ is symmetric and uniformly elliptic, then for any $\nu \in \mathcal{P}(\mathbb{R}^d)$ there exists a Feller diffusion in $\mathbb{R}^d$ with respect to L and with initial distribution ν. Furthermore, the corresponding transition measure has a λ_d–density.*

Proof See Theorem 5.11 in [Dyn65I]. □

1.3.2 Feller Diffusions in $U \subseteq \mathbb{R}^d$

The main purpose of the present chapter is to define diffusion processes in a domain $U \subseteq \mathbb{R}^d$. We have almost achieved this goal, just that so far we have only considered diffusion processes possibly taking values in the whole $\mathbb{R}^d$. In this subsection we will take the remaining step, i.e., we will consider a way to obtain Feller diffusions which live in a domain $U \subseteq \mathbb{R}^d$.

We say that $U \subseteq \mathbb{R}^d$ is a **domain** if U is connected and open in $\mathbb{R}^d$. For any domain $U \subseteq \mathbb{R}^d$ we have according to Example 4 in §26 in [Bau92] that $(U, \|\cdot\|_2)$ is a Polish space. Note that the symbol $\|\cdot\|_2$, which denotes the Euclidean norm on $\mathbb{R}^d$, also denotes the Euclidean norm on a domain in $\mathbb{R}^d$.

The following definition, which can be found in §6 of the appendix of [Dyn65II], will turn out to be essential, because frequently our considerations do not work on general domains in $\mathbb{R}^d$, but rather on bounded domains with certain properties.

Definition 1.29 *Let $n \in \mathbb{N}_0$ and $\theta > 0$.*

(i) We denote by $C^{n,\theta}(U)$ the family of all real–valued functions on U such that all partial derivatives of order n exist and are Hölder continuous with exponent θ. A function $f : U \to \mathbb{R}$ is called **Hölder continuous** *with exponent θ if there exists some $\alpha > 0$ such that $|f(x) - f(y)| \leq \alpha |x - y|^\theta$ for all $x, y \in U$.*

(ii) We call U a **$C^{n,\theta}$–domain** *if for every $y = (y_1, \ldots, y_d) \in \partial U$ there exist an $\varepsilon > 0$ and some $\varphi \in C^{n,\theta}(B((y_1, \ldots, y_{d-1}), \varepsilon))$ with*

$$B(y,\varepsilon) \cap U = B(y,\varepsilon) \cap \{x = (x_1, \ldots, x_d) \in B(y,\varepsilon) : \varphi(x_1, \ldots, x_{d-1}) < x_d\}.$$

Throughout this subsection fix some domain $U \subseteq \mathbb{R}^d$.

Let $(\acute{X}, \acute{P}^\nu)$ be the Feller diffusion in $\mathbb{R}^d$ which we have constructed in the previous subsection. Define an $\mathscr{F}^{\acute{X}}$–stopping time $\tau_U := \{t > 0 : \acute{X}_t \notin U\}$ and consider the killed process $\acute{X}^U$ given by $\acute{X}^U_t(\omega) = \acute{X}_t(\omega)$ for all $0 \leq t < \tau_U(\omega)$. Moreover, for any $\nu \in \mathcal{P}(U)$ let $\acute{P}^\nu_U \in \mathcal{P}(\Omega)$ be defined by $\acute{P}^\nu_U := \acute{P}^\nu$, and we adopt $\acute{P}^x_U := \acute{P}^{\delta_x}_U$ for each $x \in U$. The associated semigroup $(\acute{P}^U_t)_{t \in \mathbb{R}^+_0}$ of transition kernels is given by $\acute{P}^U_t(x, B) = \acute{P}^x(\acute{X}_t \in B, \tau_U > t)$ for all $t > 0$, $x \in U$ and $B \in \mathscr{B}(U)$. Note that $(\acute{P}^U_t)_{t \in \mathbb{R}^+_0}$ is a sub–Markov semigroup on $(U, \mathscr{B}(U))$.

The $\mathscr{F}^{\acute{X}}$–stopping time τ_U is referred to as the **lifetime** of $\acute{X}^U$, and we say that $\acute{X}^U$ is killed at τ_U. See also §1 of Chapter Ten in [Dyn65I] for an introduction to the notion of the killed process corresponding to a given Markov process Y. In particular, observe that by 10.2 Remark 3 in [Dyn65I] $(\acute{P}^\nu_U)_{\nu \in \mathcal{P}(U)}$ satisfies the strong Markov property. Note that in [Dyn65I] Dynkin calls the killed process "part of the process Y on the set U".

Define $\mathcal{D}_U := \{f|_U : f \in \mathcal{D} \text{ with } f \equiv 0 \text{ on } \mathbb{R}^d \backslash U\} \subseteq C_0(U)$. Then we have that the operator $T_U := T|_{\mathcal{D}_U} : \mathcal{D}_U \to C_0(U)$ is the generator of the Feller semigroup of transition operators associated with $(\acute{P}^U_t)_{t \in \mathbb{R}^+_0}$. In particular, we infer from Theorem 1.27 that T_U is given on $C^2_K(U)$ by

$$\forall\, f \in \mathcal{D}_U \,\forall\, x \in U : T_U f(x) = \frac{1}{2} \sum_{i,j=1}^d a^U_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{i=1}^d b^U_i(x) \frac{\partial}{\partial x_i} f(x) - c^U f(x),$$

where $a_{ij}^U := a_{ij}|_U, b_i^U := b_i|_U, c^U := c|_U \in C_b(U)$ for all $i, j \in \{1, \ldots, d\}$. Here the coefficients $(a_{ij})_{i,j=1,\ldots,d}$, $(b_i)_{i=1,\ldots,d}$ and c are given by Theorem 1.27. Note that the limits $\lim_{x\to\partial U} a_{ij}^U(x)$ and $\lim_{x\to\partial U} b_i^U(x)$ exist for all $i, j \in \{1, \ldots, d\}$, since there exist extensions of the respective mappings to continuous functions on $\mathbb{R}^d$. That the coefficients of T_U are restrictions of continuous maps on $\mathbb{R}^d$ to U will be relevant in Section 2.3, where we will compare the diffusions defined in Section 2.3 with the Feller diffusions in U in the sense of the following definition:

Definition 1.30 *Let $\nu \in \mathcal{P}(U)$. Then we call $(\acute{X}^U, \acute{P}_U^\nu)$ the* **Feller diffusion** *in U with respect to T_U and with initial distribution ν.*

In conclusion of this chapter we present a result which states that under certain assumptions Feller diffusions in U have nice properties in terms of a $\lambda_d|_U$–density.

Theorem 1.31 *Assume that T satisfies the assumptions of Theorem 1.28, and presume that U is a $C^{1,\theta}$–domain (cf. Definition 1.29) for some $\theta > 0$. Then the measure $\acute{P}_t^U(x, \cdot)$ has a continuous $\lambda_d|_U$–density $p_t^U(x, \cdot)$ with $\lim_{z\to x_0} p_t^U(z, y) = 0$ for all $x, y \in U$, $x_0 \in \partial U$ and $t > 0$.*

Proof See Theorem 13.18 in [Dyn65II]. □

Chapter 2

The Martingale Problem

In the previous chapter we have considered an approach to obtain Feller diffusions (cf. Definition 1.30) in some domain $U \subseteq \mathbb{R}^d$ with respect to certain linear operators. In this chapter we will develop another method, which for a particular class of linear operators leads to diffusion processes in a domain $U \subseteq \mathbb{R}^d$ which are akin to Feller diffusions in U. Our approach in this chapter is related to the approach via stochastic differential equations (SDEs). Indeed, one can show (cf. (19.7) and (20.1) Theorem in [RW00]) that a probability measure P^ν solves a given martingale problem on $\mathbb{R}^d$ iff there exists a weak solution to the corresponding SDE with distribution P^ν and initial condition $P^\nu_{X_0} = \nu$. However, it is beyond the scope of this thesis to deal with the theory of SDEs, but it is worth mentioning that the approach to obtain diffusions via weak solutions to SDEs is covered by the approach which we are going to develop in this chapter. In the first section we will cite some results by Stroock and Varadhan (cf. [SV79]) in order to introduce the martingale problem on $\mathbb{R}^d$. In Section 2.2 we will consider a localisation of the martingale problem, for that we will define the martingale problem on some domain $U \subseteq \mathbb{R}^d$. The idea of such a localisation of the martingale problem is due to Pinsky (cf. Section 13 of Chapter 1 in [Pin95]), who calles it the "generalised martingale problem on U". Finally, in Section 2.3 we will utilise the martingale problem on U in order to define diffusions. We will show that there is a strong relation between diffusions in the sense of Section 2.3 and Feller diffusions in U, as developed in Subsection 1.3.2. We would like to point out that the approach via the martingale problem is particularly useful for considerations related to the processes obtained by conditioning diffusions on not leaving U. However, in the present thesis we won't deal with such conditional processes, which have been studied e.g. by Pinsky in [Pin85] as well as by Gong, Qian and Zhao in [GQZ88]. In this respect, the present chapter has a meaning beyond its role within this thesis, because it may provide the background for further considerations in terms of conditional processes within the framework of diffusion processes. In Section 2.3 we will take up this topic again.

As suggested in the introduction above, the most relevant literature on which this chapter is based are [Pin95] and [SV79].

Throughout this chapter fix some $d \in \mathbb{N}$ and consider the Banach space $(\mathbb{R}^d, \|\cdot\|_2)$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^d$. We will also denote by the symbol $\|\cdot\|_2$ the function

$\|\cdot\|_2 : \mathscr{L}^0(\Omega) \to \mathscr{L}^0(\Omega)$, defined by $\|Y\|_2(\omega) = \|Y(\omega)\|_2$ for any $Y \in \mathscr{L}^0(\Omega)$ and all $\omega \in \Omega$, where $(\Omega, \mathscr{F})$ is some measurable space. Additionally, we presume $\mathbb{R}^d$ and any subset $U \subseteq \mathbb{R}$ to be endowed with the Borel σ–algebras $\mathscr{B}(\mathbb{R}^d)$ and $\mathscr{B}(U)$, respectively.

Let $U \subseteq \mathbb{R}^d$. We say that a second–order differential operator $T : C_K^2(U) \to \mathcal{B}(U)$ is given in **nondivergence form** if there exist some measurable locally bounded $(a_{ij})_{i,j=1,\ldots,d} : U \to \mathcal{M}_d$ and $(b_i)_{i=1,\ldots,d} : U \to \mathbb{R}^d$ such that

$$\forall\, f \in C_K^2(U)\, \forall\, x \in U : Tf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} f(x)\,.$$

Furthermore, we say that T is given in **divergence form** if there exists some differentiable $(a_{ij})_{i,j=1,\ldots,d} : U \to \mathcal{M}_d$ such that

$$\forall\, f \in C_K^2(U)\, \forall\, x \in U : Tf(x) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j} f(x)\right]$$

If moreover there exists some $(b_i)_{i=1,\ldots,d} : U \to \mathbb{R}^d$ such that

$$\forall\, f \in C_K^2(\mathbb{R}^d)\, \forall\, x \in U : Tf = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j} f(x)\right] + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} f(x),$$

then we say that the principal part of T is in divergence form. We call the matrix $(a_{ij})_{i,j=1,\ldots,d}$ **elliptic** if for every $x \in U$ there exists a $\beta_x > 0$ such that $\sum_{i,j=1}^d a_{ij}(x)\theta_i\theta_j \geq \beta_x \sum_{i=1}^d \theta_i^2$ holds for all $\theta \in \mathbb{R}^d$, and furthermore we say that $(a_{ij})_{i,j=1,\ldots,d}$ is **uniformly elliptic** if there exists some $\beta > 0$ such that $\sum_{i,j=1}^d a_{ij}(x)\theta_i\theta_j \geq \beta \sum_{i=1}^d \theta_i^2$ holds for all $x \in U$ and $\theta \in \mathbb{R}^d$.

2.1 The Martingale Problem on $\mathbb{R}^d$

Throughout this section let $(a_{ij})_{i,j=1,\ldots,d} \in C(\mathbb{R}^d, \mathcal{M}_d)$ and $(b_i)_{i=1,\ldots,d} : \mathbb{R}^d \to \mathbb{R}^d$ be measurable coefficients of a linear operator $T : C_K^2(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$ in nondivergence form, defined by

$$\forall\, f \in C_K^2(\mathbb{R}^d)\, \forall\, x \in \mathbb{R}^d : Tf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i x_j} f(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} f(x).$$

The coefficients $(a_{ij})_{i,j=1,\ldots,d}$ and $(b_i)_{i=1,\ldots,d}$ are called **diffusion coefficient** and **drift coefficient**, respectively.

We define $(\check{\Omega}, \check{\mathscr{F}}) := (C(\mathbb{R}_0^+, \mathbb{R}^d), \mathscr{B}[C(\mathbb{R}_0^+, \mathbb{R}^d)])$, where $C(\mathbb{R}_0^+, \mathbb{R}^d)$ is equipped with the topology of uniform convergence on bounded intervals of $\mathbb{R}_0^+$, and $\mathscr{B}[C(\mathbb{R}_0^+, \mathbb{R}^d)]$ denotes the Borel σ–algebra on $C(\mathbb{R}_0^+, \mathbb{R}^d)$. Furthermore, we consider the coordinate mapping process $\check{X} := (\check{X}_t)_{t \in \mathbb{R}_0^+}$ on $(\check{\Omega}, \check{\mathscr{F}})$, defined by $\check{X}_t := \pi_t : \check{\Omega} \to \mathbb{R}^d$, $t \in \mathbb{R}_0^+$, where, as in Definition B.5, π_t denotes the projection from $\check{\Omega}$ onto $\mathbb{R}^d$. Moreover, we define the filtration $\mathscr{F}^{\check{X}} := (\check{\mathscr{F}}_t)_{t \in \mathbb{R}_0^+}$, where $\check{\mathscr{F}}_t := \sigma(\check{X}_s : s \in [0, t]\}$ for any $t \in \mathbb{R}_0^+$. In addition, for any domain $D \subseteq \mathbb{R}^d$ consider the $\mathscr{F}^{\check{X}}$–stopping time $\check{\tau}_D := \inf\{t \in \mathbb{R}_0^+ : \check{X}_t \notin D\}$.

Definition 2.1 *For any $\nu \in \mathcal{P}(\mathbb{R}^d)$ we say that $P^\nu \in \mathcal{P}(\check{\Omega})$ solves the* **martingale problem** *for (T,ν) on $\mathbb{R}^d$ if*

(i) $P^\nu_{\check{X}_0} = \nu$,

(ii) $\left(f \circ \check{X}_t - \int_{[0,t]} Tf \circ \check{X}_s \lambda(ds)\right)_{t \in \mathbb{R}_0^+}$ *is an $\mathscr{F}^{\check{X}}$–martingale under P^ν for any $f \in C_K^2(\mathbb{R}^d)$.*

We say that $(P^\nu)_{\nu \in \mathcal{P}(\mathbb{R}^d)} \subseteq \mathcal{P}(\check{\Omega})$ is a solution to the martingale problem for T on $\mathbb{R}^d$ if P^ν is a solution to the martingale problem for (T,ν) on $\mathbb{R}^d$ for each $\nu \in \mathcal{P}(\mathbb{R}^d)$.

In the following we will develop results which deal with the martingale problem on $\mathbb{R}^d$, but which will turn out to be instrumental in establishing the martingale problem on a domain $U \subseteq \mathbb{R}^d$. In order to keep our proceeding reasonably concise we will start by citing some results which are proven in [SV79], and we refer the interested reader to that book for the proofs.

Theorem 2.2 *Assume that $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ are measurable and locally bounded. Furthermore, consider some measurable and locally bounded $(a'_{ij})_{i,j=1,\dots,d} : \mathbb{R}^d \to \mathcal{M}_d$ and $(b'_i)_{i=1,\dots,d} : \mathbb{R}^d \to \mathbb{R}^d$ as well as a bounded domain $U \subseteq \mathbb{R}^d$ such that $(a_{ij})_{i,j=1,\dots,d}$ and $(a'_{ij})_{i,j=1,\dots,d}$ as well as $(b_i)_{i=1,\dots,d}$ and $(b'_i)_{i=1,\dots,d}$ coincide on U. In addition, we define a linear operator $T' : C_K^2(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$ by*

$$T' := \frac{1}{2}\sum_{i,j=1}^{n} a'_{ij}\frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^{n} b'_i \frac{\partial}{\partial x_i}.$$

Fix some $\nu \in \mathcal{P}(\mathbb{R}^d)$ and presume that there exist a unique solutions P^ν to the martingale problems on $\mathbb{R}^d$ for (T,ν). Then for every solution P'^ν to the martingale problems on $\mathbb{R}^d$ for (T',ν) we have that $P^\nu = P'^\nu$ holds on $\mathscr{F}_\tau$, where τ is given by $\tau = \inf\{t \in \mathbb{R}_0^+ : \check{X}_t \notin U\}$.

Proof See 10.1.1 Theorem in [SV79]. □

Later on we will need the following variation of Tulcea's extension theorem:

Theorem 2.3 (Tulcea's Extension Theorem) *Let $(G,\mathscr{G})$ be a measurable space and let $(\mathscr{G}_n)_{n \in \mathbb{N}_0}$ be a filtration on $(G,\mathscr{G})$ with $\mathscr{G} = \bigvee_{n \in \mathbb{N}_0} \mathscr{G}_n$. Furthermore, for any $n \in \mathbb{N}_0$, let $\varphi_n : G \to \mathfrak{P}(G)$ be defined by*

$$\varphi_n(x) = \bigcap\{B \in \mathscr{G}_n : x \in B\},$$

and let μ_n be a Markov kernel from $(G,\mathscr{G}_{n-1})$ to $(G,\mathscr{G}_n)$ such that $\mu_n(x,B) = 0$ holds for all $B \in \mathscr{G}_n$ with $\varphi_{n-1}(x) \cap B = \emptyset$. If, moreover, we have that $\bigcap_{n \in \mathbb{N}_0} \varphi_n(x_n) \neq \emptyset$ holds true for any sequence $(x_n)_{n \in \mathbb{N}_0}$ in G satisfying $\bigcap_{n=0}^N \varphi_n(x_n) \neq \emptyset$ for all $N \in \mathbb{N}$, then for every probability measure ν on $(G,\mathscr{G}_0)$ there exists a uniquely defined probability measure P^ν on $(G,\mathscr{G})$ agreeing with ν on $\mathscr{G}_0$ and such that

$$\forall\, n \in \mathbb{N}\, \forall\, B \in \mathscr{G}_n : P^\nu(B) = \int_G \mu_n(\cdot, B)\, dP^\nu.$$

Proof Confer 1.1.9 Theorem in [SV79]. □

Theorem 2.4 *Presume that* $(a_{ij})_{i,j=1,\dots,d} \in C_b(\mathbb{R}^d, \mathcal{M}_d)$ *is elliptic and* $(b_i)_{i=1,\dots,d} : \mathbb{R}^d \to \mathbb{R}^d$ *is measurable and bounded. Then the martingale problem for* T *on* $\mathbb{R}^d$ *has a solution which is unique.*

Proof See 7.2.1 Theorem in [SV79]. □

Theorem 2.5 *Assume that* $(a_{ij})_{i,j=1,\dots,d}$ *and* $(b_i)_{i=1,\dots,d}$ *are as in Theorem 2.4. Furthermore, let* $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ *denote the unique solution to the martingale problem for* T *on* $\mathbb{R}^d$*. Then a family* $\mathscr{P} \subseteq (P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ *is relatively weakly compact if*

$$\lim_{k\to\infty} \sup_{P\in\mathscr{P}} P(\|\check{X}_0\|_2 \geq k) = 0. \tag{2.1}$$

Proof The idea is to apply 1.4.6 Theorem in [SV79], i.e., we have to show that for all nonnegative $f \in C_K^\infty(\mathbb{R}^d)$ there exists a constant $\alpha_f \geq 0$ such that $(f\circ \check{X}_t + \alpha_f t)_{t\in\mathbb{R}_0^+}$ is an $\mathscr{F}^{\check{X}}$–martingale under any $P \in \mathscr{P}$, and that α_f works for all translates of f.

Let $f \in C_K^\infty(\mathbb{R}^d)$. Because each $P \in \mathscr{P}$ is the solution to the martingale problem for (T,ν) on $\mathbb{R}^d$ for some $\nu \in \mathcal{P}(\mathbb{R}^d)$, we obtain

$$\begin{aligned}\int_A (f\circ\check{X}_t - f\circ X_s)\,\mathrm{d}P &= \int_A\int_{(s,t]} Tf\circ\check{X}_u\lambda(\mathrm{d}t)\,\mathrm{d}P \\ &\geq (t-s)\min_{x\in\mathbb{R}^d} Tf(x)P(A) \\ &\geq \begin{cases} 0, & \min_{x\in\mathbb{R}^d} Tf(x)\geq 0 \\ -\min_{x\in\mathbb{R}^d} Tf(x)P(A)(s-t), & \min_{x\in\mathbb{R}^d} Tf(x) < 0\end{cases}\end{aligned} \tag{2.2}$$

for all $s,t\in\mathbb{R}_0^+$ with $s\leq t$, any $A \in \check{\mathscr{F}}_s$ and every $P\in\mathscr{P}$. Note that $\min_{x\in\mathbb{R}^d} Tf(x)$ exists, since $Tf \in \mathcal{B}(\mathbb{R}^d)$. If $\min_{x\in\mathbb{R}^d} Tf(x) \geq 0$, choose $\alpha_f := 0$. Otherwise, if $\min_{x\in\mathbb{R}^d} Tf(x) < 0$, let $\alpha_f := -\min_{x\in\mathbb{R}^d} Tf(x)$. Now we deduce from (2.2) that in both cases $(f\circ\check{X}_t + \alpha_f t)_{t\in\mathbb{R}_0^+}$ is an $\mathscr{F}^{\check{X}}$–submartingale under any $P\in\mathscr{P}$. Moreover, such an $\mathscr{F}^{\check{X}}$–submartingale property under every $P\in\mathscr{P}$ is satisfied with the same α_f for any translate of f, because $Tc = 0$ for all $c\in\mathbb{R}$. By means of 1.4.6 Theorem in [SV79] this yields the assertion. □

Lemma 2.6 *Presume that* $(a_{ij})_{i,j=1,\dots,d}$ *and* $(b_i)_{i=1,\dots,d}$ *are as in Theorem 2.4. In addition, let* $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ *denote the unique solution to the martingale problem for* T *on* $\mathbb{R}^d$*, which exists according to Theorem 2.4. Then* $P^{(\cdot)}(A) : \mathbb{R}^d \to [0,1]$*,* $x \mapsto P^x(A)$*, is* $\mathscr{B}(\mathbb{R}^d)$*–*$\mathscr{B}([0,1])$*–measurable for each* $A \in \check{\mathscr{F}}$*.*

Proof See 6.7.4 in [SV79]. □

Theorem 2.7 *Assume that* $(a_{ij})_{i,j=1,\dots,d}$ *and* $(b_i)_{i=1,\dots,d}$ *satisfy the assumptions of Theorem 2.4. Furthermore, let* $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ *denote the unique solution to the martingale problem for* T *on* $\mathbb{R}^d$*. Then* $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ *satisfies the strong Markov property. If moreover* $(b_i)_{i=1,\dots,d}$ *is continuous, then* $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ *satisfies the Feller property.*

Proof These properties can be shown on the lines of the proofs that the unique solution to the martingale problem for T on a domain $U \subseteq \mathbb{R}^d$ (cf. Section 2.2), if existent, satisfies these properties. For this purpose see Theorem 2.23 on page 36 and Theorem 2.24 on page 38. □

Theorem 2.8 *Presume that $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ satisfy the assumptions of Theorem 2.4, and let $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ denote the unique solution to the martingale problem for T on $\mathbb{R}^d$. Then $\int_{\check\Omega} Y\ dP^{(\cdot)}$ is continuous on $\mathbb{R}^d$ for any $Y \in \mathcal{B}(\check\Omega)$ which is P^y–a.s. continuous for every $y \in \mathbb{R}^d$.*

Proof Let $x \in \mathbb{R}^d$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ such that $x_n \to x$ as $n \to \infty$. Then the Feller property of $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ implies that $P^{x_n}_{X_t} \to P^x_{X_t}$, $t \in \mathbb{R}_0^+$, weakly as $n \to \infty$. At first we show that

$$\forall\, m \in \mathbb{N}\,\forall\,(t_1,\dots,t_m) \in \mathcal{H}(\mathbb{R}_0^+) : P^{x_n}_{(X_{t_1},\dots,X_{t_m})} \to P^x_{(X_{t_1},\dots,X_{t_m})} \tag{2.3}$$

weakly as $n \to \infty$. By means of the Markov property of $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ and Lemma 1.8 we obtain for any $f \in C_K(\mathbb{R}^{md})$ that

$$\begin{aligned}
&\int_{\mathbb{R}^{md}} f\ \mathrm{d}P^{x_n}_{(X_{t_1},\dots,X_{t_m})}\\
=\ &\int_{\mathrm{supp}\,(f)} \cdots \int_{\mathrm{supp}\,(f)} f(y_1,\dots,y_m) P^{y_{m-1}}_{X_{t_m}}(\mathrm{d}y_m) P^{y_{m-2}}_{X_{t_{m-1}}}(\mathrm{d}y_{m-1}) \dots P^{y_1}_{X_{t_2}}(\mathrm{d}y_2) P^{x_n}_{X_{t_1}}(\mathrm{d}y_1)\\
\overset{(*)}{\to}\ &\int_{\mathrm{supp}\,(f)} \cdots \int_{\mathrm{supp}\,(f)} f(y_1,\dots,y_m) P^{y_{m-1}}_{X_{t_m}}(\mathrm{d}y_m) P^{y_{m-2}}_{X_{t_{m-1}}}(\mathrm{d}y_{m-1}) \dots P^{y_1}_{X_{t_2}}(\mathrm{d}y_2) P^{x}_{X_{t_1}}(\mathrm{d}y_1)\\
=\ &\int_{\mathbb{R}^{md}} f\ \mathrm{d}P^{x}_{(X_{t_1},\dots,X_{t_m})}
\end{aligned}$$

as $n \to \infty$. Note that it is sufficient to consider $f \in C_K(\mathbb{R}^{md})$, since on $\mathcal{P}(\mathbb{R}^d)$ the concepts of weak convergence and vague convergence coincide (cf. p. 4). In order to prove $(*)$ we show that $\int_K f(\cdot,y)P^{(\cdot)}(\mathrm{d}y)$ is continuous on K, where $f \in C_K(\mathbb{R}^{2d})$ with compact support $K \subseteq \mathbb{R}^d$. Let $\varepsilon > 0$. To begin with, observe that $\int_K f(v,y)P^{(\cdot)}(\mathrm{d}y)$ is continuous for every $v \in K$, i.e.,

$$\forall\, z \in K\ \exists\, \delta_1 > 0\ \forall\, z' \in B[z,\delta_1] : \left|\int_K f(v,y)P^z(\mathrm{d}y) - \int_K f(v,y)P^{z'}(\mathrm{d}y)\right| \le \frac{\varepsilon}{3}. \tag{2.4}$$

Since f is continuous and K^2 is compact, we infer that f is uniformly continuous on K^2. In view of the uniform continuity of f on K^2, choose some $\delta_2 > 0$ such that

$$|f(z,y) - f(v,y)| \le \varepsilon/3 \tag{2.5}$$

for all $z, v \in K$ with $\|z - v\|_2 \le \delta_2$. Fix $z \in K$ and let $z' \in B[z, \min(\delta_1, 2\delta_2)]$. In addition, choose some $v \in B[z,\delta_2] \cap B[z',\delta_2]$. Then we have

$$\left|\int_K f(z,y)P^z(\mathrm{d}y) - \int_K f(z',y)P^{z'}(\mathrm{d}y)\right|$$

$$\begin{aligned}
&= \left| \int_K [f(z,y) - f(v,y)] P^z(\mathrm{d}y) + \int_K f(v,y) P^z(\mathrm{d}y) \right. \\
&\qquad \left. - \int_K f(v,y) P^{z'}(\mathrm{d}y) - \int_K [f(z',y) - f(v,y)] P^{z'}(\mathrm{d}y) \right| \\
&\leq \int_K |f(z,y) - f(v,y)| P^z(\mathrm{d}y) + \int_K |f(z',y) - f(v,y)| P^{z'}(\mathrm{d}y) \\
&\qquad + \left| \int_K f(v,y) P^z(\mathrm{d}y) - \int_K f(v,y) P^{z'}(\mathrm{d}y) \right| \\
&\underset{(2.5)}{\overset{(2.4)}{\leq}} \varepsilon
\end{aligned}$$

which shows that $\int_K f(\cdot, y) P^{(\cdot)}(\mathrm{d}y)$ is continuous on K. Thus, $(*)$ is established.

We want to apply Theorem 2.5 in order to deduce that $(P^{x_n})_{n\in\mathbb{N}}$ is relatively weakly compact. To this end we have to show that (2.1) is satisfied. For this purpose choose some $\varepsilon > 0$. Since $x_n \to x$, there exists an $n_0 \in \mathbb{N}$ such that $\|x - x_n\|_2 \leq \varepsilon$ for all $n \geq n_0$. With $k > \max(\max\{\|x_n\|_2 : n < n_0\}, \|x\|_2 + \varepsilon)$ we obtain that $\sup_{n\in\mathbb{N}} P^{x_n}(\|\check{X}_0\|_2 \geq k) = 0$, which shows that (2.1) is satisfied. Consequently, by means of Theorem 2.5, the sequence $(P^{x_n})_{n\in\mathbb{N}}$ is relatively weakly compact. Hence, 7.8 Theorem (b) in Chapter 3 in [EK86] in conjunction with (2.3) yields that $P^{x_n} \to P^x$ weakly as $n \to \infty$. Therefore, $\int_{\check{\Omega}} Y \,\mathrm{d}P^{(\cdot)}$ is continuous on $\mathbb{R}^d$ for all $Y \in C_b(\check{\Omega})$. Now consider some $Y \in \mathcal{B}(\check{\Omega})$ which is P^y–a.s. continuous for every $y \in \mathbb{R}^d$, i.e.,

$$\forall\, y \in \mathbb{R}^d : P^y(\omega \in \check{\Omega} : \omega \text{ is a point of discontinuity of } Y) = 0.$$

Futhermore, put $A := \{\omega \in \check{\Omega} : Y \text{ is continuous at } \omega\}$. Then $P^y(A) = 1$ for each $y \in \mathbb{R}^d$, and thus

$$\int_{\check{\Omega}} Y \,\mathrm{d}P^{x_n} = \int_A Y \,\mathrm{d}P^{x_n} \to \int_A Y \,\mathrm{d}P^x = \int_{\check{\Omega}} Y \,\mathrm{d}P^x$$

as $n \to \infty$, which proves the assertion. □

Theorem 2.9 *Assume that $(a_{ij})_{i,j=1,\ldots,d}$ and $(b_i)_{i=1,\ldots,d}$ satisfy the assumptions of Theorem 2.4, and denote the unique solution to the martingale problem for T on $\mathbb{R}^d$ by $(P^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$. Then $P^\nu_{\check{X}_t}$ possesses a λ_d–density for any $\nu \in \mathcal{P}(\mathbb{R}^d)$.*

Proof Confer 9.2.2 Lemma in [SV79]. □

Now we have compiled all the results which will turn out to be adjuvant in Section 2.2, where we will extend the theory compiled above in order to establish more general results. Much of the following section is based upon Chapter 1 in [Pin95]. However, our aim is to present that theory within our framework, i.e., we will focus on the results related to the martingale problem on a domain in $\mathbb{R}^d$. In [Pin95] Pinsky states most of the results which we are going to develop, but often he only proves them in a special case or he doesn't give a proof at all. Even though the main ideas of the proofs are similar, they have to be modified considerably in order to apply them in our situation. Moreover, we will give more detailed proofs than the ones in [Pin95].

2.2 The Martingale Problem on $U \subseteq \mathbb{R}^d$

In this section we will consider a localisation of the the theory compiled in the previous section. This approach has the benefit that we can relax the assumptions regarding the boundedness of $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$.

Throughout the whole section fix some domain $U \subsetneq \mathbb{R}^d$. Note that we allow U to be unbounded. Furthermore, let $(a_{ij})_{i,j=1,\dots,d} \in C(U, \mathcal{M}_d)$ and $(b_i)_{i=1,\dots,d} : U \to \mathbb{R}^d$ be locally bounded measurable coefficients of an operator $T : C_K^2(U) \to \mathcal{B}(U)$ defined by

$$T := \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} .$$

Let us denote by $\Delta \in \mathbb{R}^d \setminus U$ a point which yields an Alexandroff one–point compactification $\hat{U} := U \cup \{\Delta\}$ of U (cf. Theorem B.24). In the literature Δ is sometimes called "point at infinity" or "cemetery state". Let $\mathcal{M}$ be a basis of neighbourhoods at Δ, where the elements of $\mathcal{M}$ are the complements in $\hat{U}$ of the compact sets in U. By Example 6 in §26 in [Bau92] the compact space $(\hat{U}, \mathcal{T}_{\hat{U}})$ is a Polish space. The topology $\mathcal{T}_{\hat{U}}$ is induced by the metric $\rho_{\hat{U}}$ on $\hat{U} \times \hat{U}$ which is given by

$$\forall\, x, y \in \hat{U} : \rho_{\hat{U}}(x, y) = \begin{cases} \inf_{z \in \partial U} \|x - z\|_2, & x \in U, y = \Delta \\ \min(\|x - y\|_2, \rho_{\hat{U}}(x, \Delta) + \rho_{\hat{U}}(y, \Delta)), & x, y \in U. \end{cases}$$

Consider the function $\delta : U \to \mathbb{R}^+$ given by $\delta(x) = \inf_{y \in \partial U} \|x - y\|_2$ for all $x \in U$. Observe that $\delta(x) = \rho_{\hat{U}}(x, \Delta)$ for all $x \in U$.

Throughout the whole section let $(U_n)_{n \in \mathbb{N}}$ be a sequence of bounded domains in U with $U = \bigcup_{n \in \mathbb{N}} U_n$ and such that $\bar{U}_n \subsetneq U_{n+1}$ for all $n \in \mathbb{N}$. We define functions $\tau_n, \tau_U' : \hat{U}^{\mathbb{R}_0^+} \to \bar{\mathbb{R}}_0^+$, $n \in \mathbb{N}$, by $\tau_n'(f) = \inf\{t \in \mathbb{R}_0^+ : f(t) \notin U_n\}$ and $\tau_U'(f) = \inf\{t \in \mathbb{R}_0^+ : f(t) \notin U\}$ for all $n \in \mathbb{N}$ and $f \in \hat{U}^{\mathbb{R}_0^+}$. Let us adopt

$$\Omega := \{\omega \in C(\mathbb{R}_0^+, \hat{U}) : \omega(t) = \Delta \,\forall\, t \geq \tau_U'(\omega)\}$$

and $\tilde{\Omega} :=$

$$\{\omega \in \hat{U}^{\mathbb{R}_0^+} : [\omega|_{[0,\tau_U'(\omega))} \in C([0, \tau_U'(\omega)), U)] \wedge [\omega(t) = \Delta \,\forall\, t \geq \tau_U'(\omega)] \wedge [\tau_n'(\omega) < \tau_U'(\omega) \,\forall\, n \in \mathbb{N}]\}.$$

Below we will point out briefly why we consider $\tilde{\Omega}$.

Here presume Ω to be endowed with the topology of uniform convergence on bounded intervalls of $\mathbb{R}_0^+$, denoted by $\mathcal{T}$. Note that $(\Omega, \mathcal{T})$ is a Polish space. Let $X := (X_t)_{t \in \mathbb{R}_0^+}$ and $\tilde{X} := (\tilde{X}_t)_{t \in \mathbb{R}_0^+}$ be defined by $X_t := \pi_t : \Omega \to \hat{U}$ and $\tilde{X}_t := \pi_t : \tilde{\Omega} \to \hat{U}$ for every $t \in \mathbb{R}_0^+$. Then we get measurable spaces $(\Omega, \mathscr{F})$ and $(\tilde{\Omega}, \tilde{\mathscr{F}})$, where $\mathscr{F} := \mathscr{B}(\Omega)$ and $\tilde{\mathscr{F}} := \sigma(\tilde{X})$. Here $\mathscr{B}(\Omega)$ denotes the Borel σ–algebras on Ω. Moreover, for any $t \in \mathbb{R}_0^+$ we adopt $\mathscr{F}_t := \sigma(X_s : s \in [0, t])$ and $\tilde{\mathscr{F}}_t := \sigma(\tilde{X}_s : s \in [0, t])$, and therewith we obtain filtrations

$\mathscr{F}^X := (\mathscr{F}_t)_{t\in\mathbb{R}_0^+}$ as well as $\mathscr{F}^{\tilde{X}} := (\tilde{\mathscr{F}}_t)_{t\in\mathbb{R}_0^+}$. Furthermore, we have $\mathscr{F} = \sigma(X) = \bigvee_{t\in\mathbb{R}_0^+} \mathscr{F}_t$. For every domain $D \subseteq \mathbb{R}^d$ let us define an $\mathscr{F}^X$–stopping time $\tau_D := \inf\{t \in \mathbb{R}_0^+ : X_t \notin D\}$ as well as an $\mathscr{F}^{\tilde{X}}$–stopping time $\tilde{\tau}_D := \inf\{t \in \mathbb{R}_0^+ : \tilde{X}_t \notin D\}$.

Later on we will only be concerned with Ω, since the diffusions which we are going to define have trajectories in Ω. But in order to prove the main result of this chapter, we will show that for any $\nu \in \mathcal{P}(U)$ there exists a probability measure $P^\nu \in \mathcal{P}(\tilde{\Omega})$ with the desired properties, and then, in order to complete the proof of that theorem, we will prove that $P^\nu(\tilde{\Omega} \setminus \Omega) = 0$. The reason why we have to deal with $\tilde{\Omega}$ is that a map $\omega \in \tilde{\Omega}$ may not have a limit from the left as $t \uparrow \tau_U(\omega)$, whereas this is not possible for functions in Ω. This difference will be of importance, and we will explain it in more detail in Remark 2.20 on page 36. The underlying idea is somewhat similar to the idea of the proof of the Kolomogorov extension theorem (cf. Theorem B.8), where one also uses a larger space. We will apply Tulcea's extension theorem (cf. Theorem 2.3). Note that Tulcea's extension theorem does not require that the underlying space has a metric, and thus we don't have to specify a topology on $\tilde{\Omega}$.

For any family $(P^\nu)_{\nu\in\mathcal{P}(U)}$ of probability measures on $(\Omega, \mathscr{F})$ and every $\nu \in \mathcal{P}(U)$ we denote by E_ν the expectation with respect to P^ν. As in the previous chapter we adopt $P^x := P^{\delta_x}$ and $E_x := E_{\delta_x}$ for all $x \in U$. In addition, for any $t \in \mathbb{R}_0^+$ and each $A \in \mathscr{F}$ we define the random variable $P^{X_t}(A) := P^{(\cdot)}(A) \circ X_t$. Analogous adoptions will be valid if we replace $(\Omega, \mathscr{F})$ by $(\tilde{\Omega}, \tilde{\mathscr{F}})$.

Presume that $(a_{ij})_{i,j=1,\dots,d}$ is elliptic. For every $n \in \mathbb{N}$ let us consider a cut–off function $\psi_n \in C_K^\infty(\mathbb{R}^d, [0,1])$ with $\psi_n \equiv 1$ on U_n and $\psi_n \equiv 0$ on $\mathbb{R}^d \setminus U_{n+1}$. Using $(\psi_n)_{n\in\mathbb{N}}$ we can construct sequences $(a^n)_{n\in\mathbb{N}}$ and $(b^n)_{n\in\mathbb{N}}$ of functions from $\mathbb{R}^d$ to $\mathbb{R}$ as follows:

$$\forall\, n \in \mathbb{N}\, \forall\, i,j \in \{1,\dots,d\} : a_{ij}^n := a'_{ij} \cdot \psi_n + (\mathbf{1} - \psi_n)\delta_{ij} \quad \text{and} \quad b_i^n := b'_i \cdot \psi_n, \tag{2.6}$$

where $a'_{ij}, b'_i : \mathbb{R}^d \to \mathbb{R}$ are given by $a'_{ij} = a_{ij}\mathbb{1}_U$ and $b'_i = b_i \mathbb{1}_U$ for all $i,j \in \{1,\dots,d\}$. Observe that (2.6) yields that $(a_{ij}^n)_{i,j=1,\dots,d}$ is bounded, continuous as well as elliptic, and $(b_i^n)_{i=1,\dots,d}$ is bounded and continuous for all $n \in \mathbb{N}$, i.e., $(a_{ij}^n)_{i,j=1,\dots,d}$ and $(b_i^n)_{i=1,\dots,d}$ satisfy the assumptions of Theorem 2.4. For any $n \in \mathbb{N}$ we define a linear operator $T_n : C_K^2(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$ by

$$T_n := \frac{1}{2}\sum_{i,j=1}^n a_{ij}^n \frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^n b_i^n \frac{\partial}{\partial x_i}. \tag{2.7}$$

By means of Theorem 2.4 there exits a unique solution $(P_n^\nu)_{\nu\in\mathcal{P}(\mathbb{R}^d)}$ to the martingale problem for T_n on $\mathbb{R}^d$. We denote the expectation with respect to P_n^ν, $\nu \in \mathcal{P}(\mathbb{R}^d)$, by E_ν^n. Throughout this section terms as P_n^ν will always refer to the aforementioned solution, which will play a crucial role for our considerations.

For each $n \in \mathbb{N}$ let us define an $\mathscr{F}^X$–stopping time $\tau_n := \inf\{t \in \mathbb{R}_0^+ : X_t \notin U_n\}$ as well as an $\mathscr{F}^{\tilde{X}}$–stopping time $\tilde{\tau}_n := \inf\{t \in \mathbb{R}_0^+ : \tilde{X}_t \notin U_n\}$.

Throughout this section we consider a family of stochastic processes defined by

$$\forall\, n \in \mathbb{N}\, \forall\, f \in C_K^2(U)\, \forall\, t \in \mathbb{R}_0^+ : M_t^{n,f} := f \circ X_{t\wedge\tau_n} - \int_{[0,t\wedge\tau_n]} Tf \circ X_s \lambda(\mathrm{d}s).$$

Definition 2.10 *Let $\nu \in \mathcal{P}(U)$, then a probability measure $P^\nu \in \mathcal{P}(\Omega)$ is said to solve the* **martingale problem** *for (T,ν) on U if*

(i) $P^\nu_{X_0} = \nu$,

(ii) $(M_t^{n,f})_{t\in\mathbb{R}_0^+}$ *is an $\mathscr{F}^X$–martingale under P^ν for any $n \in \mathbb{N}$ and every $f \in C_K^2(U)$.*

We say that $(P^\nu)_{\nu\in\mathcal{P}(U)} \subseteq \mathcal{P}(\Omega)$ is a solution to the martingale problem for T on U if P^ν is a solution to the martingale problem for (T,ν) on U for each $\nu \in \mathcal{P}(U)$.

We will occasionally utilise the following lemma, which can be found in [SV79].

Lemma 2.11 *For every $\mathscr{F}^X$–stopping time τ as well as any $\mathscr{F}^{\tilde{X}}$–stopping time $\tilde{\tau}$ we have $\mathscr{F}_\tau = \sigma(X_{t\wedge\tau} : t \in \mathbb{R}_0^+)$ and $\tilde{\mathscr{F}}_{\tilde{\tau}} = \sigma(\tilde{X}_{t\wedge\tilde{\tau}} : t \in \mathbb{R}_0^+)$.*

Proof See 1.3.3 Lemma in [SV79]. Note that there the assertion is shown for $C(\mathbb{R}_0^+, \mathbb{R}^d)$ instead of Ω and $\tilde{\Omega}$, respectively. However, the proof does not rely on that, and thus it works out equally well with Ω and $\tilde{\Omega}$. □

Remark 2.12 In particular, Lemma 2.11 yields that $\bigvee_{n\in\mathbb{N}} \mathscr{F}_{\tau_n} = \mathscr{F}_{\tau_U} = \mathscr{F}_{\tau_U\vee t} = \mathscr{F}$ and $\bigvee_{n\in\mathbb{N}} \tilde{\mathscr{F}}_{\tilde{\tau}_n} = \tilde{\mathscr{F}}_{\tilde{\tau}_U} = \tilde{\mathscr{F}}_{\tilde{\tau}_U\vee t} = \tilde{\mathscr{F}}$ hold true for every $t \in \mathbb{R}_0^+$. The reason is that $\sigma(X_{\tau_U}) = \sigma(X_{\tau_U+t} : t \in \mathbb{R}_0^+) = \{\emptyset, \Omega\}$ and $\sigma(\tilde{X}_{\tilde{\tau}_U}) = \sigma(\tilde{X}_{\tilde{\tau}_U+t} : t \in \mathbb{R}_0^+) = \{\emptyset, \tilde{\Omega}\}$, because $X_{\tau_U+t} = \tilde{X}_{\tilde{\tau}_U+t} \equiv \Delta$ for all $t \in \mathbb{R}_0^+$. ◊

For every $\nu \in \mathcal{P}(U)$ we denote by $\check{\nu} \in \mathcal{P}(\mathbb{R}^d)$ the corresponding extended measure defined by $\check{\nu}|_{\mathscr{B}(U)} := \nu$ and $\check{\nu}(\mathbb{R}^d \setminus U) := 0$.

Define $\Psi : \check{\Omega} \to \tilde{\Omega}$ by $\Psi(\omega) = \omega \mathbb{1}_{[0,\tau_U(\omega))} + \Delta \mathbb{1}_{[\tau_U(\omega),\infty)}$ for all $\omega \in \check{\Omega}$. Observe that Ψ is $\check{\mathscr{F}}_{\tilde{\tau}_n}$–$\tilde{\mathscr{F}}_{\tilde{\tau}_n}$–measurable for all $n \in \mathbb{N}$, since $\Psi^{-1}(\tilde{X}^{-1}_{t\wedge\tilde{\tau}_n}(B)) = \check{X}^{-1}_{t\wedge\tilde{\tau}_n}(B)$ for all $t \in \mathbb{R}_0^+$ and $B \in \mathscr{B}(U)$, and because we have $\check{\mathscr{F}}_{\tilde{\tau}_n} = \sigma(\check{X}^{-1}_{t\wedge\tilde{\tau}_n}(B) : t \in \mathbb{R}_0^+, B \in \mathscr{B}(U))$ as well as $\tilde{\mathscr{F}}_{\tilde{\tau}_n} = \sigma(\tilde{X}^{-1}_{t\wedge\tilde{\tau}_n}(B) : t \in \mathbb{R}_0^+, B \in \mathscr{B}(U))$. For any $n \in \mathbb{N}$ and $A \in \tilde{\mathscr{F}}_{\tilde{\tau}_n}$ let us adopt $\check{A} := \Psi^{-1}(A) \in \check{\mathscr{F}}_{\tilde{\tau}_n}$. Analogously we obtain that Ψ is $\check{\mathscr{F}}_{\tilde{\tau}_n}$–$\mathscr{F}_{\tau_n}$–measurable for all $n \in \mathbb{N}$, and we adopt $\check{A} := \Psi^{-1}(A)$ for any $A \in \mathscr{F}_{\tau_n}$. These adoptions will be valid throughout the whole section.

With the set–up developed above, which will turn out to be essential for our further considerations, we obtain the following lemma which constitutes an indispensable tool for our proceeding.

Lemma 2.13 *Fix some arbitrary $\nu \in \mathcal{P}(U)$ and assume that there exists a $P^\nu \in \mathcal{P}(\Omega)$ with*

$$\forall\, n \in \mathbb{N}\, \forall\, A \in \mathscr{F}_{\tau_n} : P^\nu(A) = P_n^{\check{\nu}}(\check{A}). \tag{2.8}$$

Then the following properties hold:

(i) P^ν is uniquely determined by (2.8).

(ii) P^ν is a solution to the martingale problem for (T,ν) on U.

Proof (i) Fix some $t \in \mathbb{R}_0^+$ and let $A \in \mathscr{F}_t$. Furthermore, we set $A_U := A \cap \{\tau_U \le t\}$, and for every $n \in \mathbb{N}$ we put $A_n := A \cap \{\tau_n > t\}$. According to Remark 2.12 we have $\mathscr{F}_{\tau_U} = \mathscr{F}_{\tau_U \vee t}$, and thus $A \cap \{\tau_U \le t\} \in \mathscr{F}_{\tau_U}$. Moreover, according to Remark 2.12 we have $\bigvee_{n\in\mathbb{N}} \mathscr{F}_{\tau_n} = \mathscr{F}_{\tau_U}$. In view of (2.8) $P^\nu(A_U)$ is therefore uniquely determined by the family $(P_n^{\check{\nu}})_{n\in\mathbb{N}}$. Now we show that $P^\nu(A \cap \{\tau_U > t\})$ is uniquely determined by (2.8). To this end we infer from the upper continuity of the measure P^ν and (2.8) that

$$\begin{aligned} P^\nu(A \cap \{\tau_U > t\}) &= \lim_{n\to\infty} P^\nu(A \cap \{\tau_n \le t\} \cap \{\tau_U > t\}) + \lim_{n\to\infty} P^\nu(A \cap \{\tau_n > t\}) \\ &= \lim_{n\to\infty} P^\nu(A_n) = \lim_{n\to\infty} P_n^{\check{\nu}}(\check{A}_n), \end{aligned}$$

because $A \cap \{\tau_n > t\} \in \mathscr{F}_t$, and thus $A \cap \{\tau_n > t\} \in \mathscr{F}_{\tau_n}$, for all $n \in \mathbb{N}$. Hence we conclude that $P^\nu(A)$ is uniquely determined by (2.8), which yields the assertion, since $\mathscr{F} = \bigvee_{t\in\mathbb{R}_0^+} \mathscr{F}_t$.

(ii) Fix some $n \in \mathbb{N}$ as well as $f \in C_K^\infty(U)$, and observe that $P_{X_0}^\nu(B) = P_n^{\check{\nu}}(\check{X}_0 \in B) = \nu(B)$ holds true by (2.8) for all $B \in \mathscr{B}(U)$. Since $(f \circ \check{X}_t - \int_{[0,t]} T_n f \circ \check{X}_s \lambda(\mathrm{d}s))_{t\in\mathbb{R}_0^+}$ is an $\mathscr{F}^{\check{X}}$–martingale under $P_n^{\check{\nu}}$, we infer from the Optional Sampling Theorem (cf. Theorem B.18) that $(f \circ \check{X}_{t\wedge\check{\tau}_n} - \int_{[0,t\wedge\check{\tau}_n]} T_n f \circ \check{X}_s \lambda(\mathrm{d}s))_{t\in\mathbb{R}_0^+}$ is an $\mathscr{F}^{\check{X}}$–martingale under $P_n^{\check{\nu}}$, where $\check{\tau}_n := \inf\{t \in \mathbb{R}_0^+ : \check{X}_t \notin U_n\}$. Note that we can indeed apply the Optional Sampling Theorem, because $t \wedge \check{\tau}_n \in \mathcal{S}_b(\mathscr{F}^{\check{X}})$ for all $t \in \mathbb{R}_0^+$. Moreover, note that $Tf = T_n f$ on $\bar{U}_n$, because $a = a^n$ and $b = b^n$ on $\bar{U}_n$. Therefore, it follows from (2.8) that $(f \circ X_{t\wedge\tau_n} - \int_{[0,t\wedge\tau_n]} Tf \circ X_s \lambda(\mathrm{d}s))_{t\in\mathbb{R}_0^+}$ is an $\mathscr{F}^X$–martingale under P^ν, which proves the assertion. Note that here we have used that $\tau_n(\omega) = \check{\tau}_n(\check{\omega})$ for all $\omega \in \Omega$ and $\check{\omega} \in \check{\Omega}$ with $\omega = \check{\omega}$ on $[0, \tau_n(\omega)]$.

□

Remark 2.14 Let $\nu \in \mathcal{P}(U)$, and observe that by Theorem 2.2 every solution to the martingale problem for (T, ν) on U satisfies (2.8), since $a_{ij} = a_{ij}^n$ and $b_i = b_i^n$ on U_n for all $n \in \mathbb{N}$ and $i, j \in \{1, \dots, d\}$. Therefore, we infer from Lemma 2.13 (ii) that a measure $P^\nu \in \mathcal{P}(\Omega)$ is a solution to the martingale problem for (T, ν) on U iff it satisfies (2.8). In conjunction with Lemma 2.13 (i) this yields in particular that there exists at most one solution to the martingale problem for (T, ν) on U. ◊

Let $(\check{a}_{ij})_{i,j=1,\dots,d} : \mathbb{R}^d \to \mathcal{M}_d$ be bounded, continuous as well as elliptic, and let $(\check{b})_{i=1,\dots,d} : \mathbb{R}^d \to \mathbb{R}^d$ be bounded as well as continuous. Consider the linear operator $\check{T} : C_K^2(\mathbb{R}^d) \to C_K(\mathbb{R}^d)$ defined by $\check{T} := \frac{1}{2}\sum_{i,j=1}^d \check{a}_{ij} \frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^d \check{b}_i \frac{\partial}{\partial x_i}$ and let $x \in \mathbb{R}^d$. Then the martingale problem for $(\check{T}, \delta_x)$ on $\mathbb{R}^d$ has a unique solution $\check{P}^x \in \mathcal{P}(\check{\Omega})$.

In the proof of the main theorem in this chapter we will utilise the following two theorems which can both be found in [Pin95].

Theorem 2.15 *Let $x \in \mathbb{R}^d$ and let $B \subseteq \mathbb{R}^d$ be a domain satisfying an exterior cone condition at any $y \in \partial B$, i.e., for any $y \in \partial B$ there exists a cone in $\mathbb{R}^d \setminus B$ with base at y. Then $\check{\tau}_B$ is a $\check{P}^x$–a.s. continuous function from $\check{\Omega} \to \bar{\mathbb{R}}_0^+$.*

Proof Confer Theorem 3.3 (iv) in Chapter 2 in [Pin95]. □

Remark 2.16 Let $x \in U$ and $\alpha > 0$. Since $B(x,\alpha)$ satisfies an exterior cone condition at any $y \in \partial B(y,\alpha)$, we infer from Theorem 2.15 that the mapping $\check{\tau}_{B(x,\alpha)} : \check{\Omega} \to \bar{\mathbb{R}}_0^+$ is $\check{P}^x$–a.s. continuous. $\diamondsuit$

Theorem 2.17 (Stroock-Varadhan Support Theorem) *Let $x \in \mathbb{R}^d$, then we have*

$$\operatorname{supp}(\check{P}^x) = \{\omega \in \check{\Omega} : \check{X}_0(\omega) = x\}.$$

Proof See Theorem 6.1 in Chapter 2 in [Pin95]. $\square$

Later on, the Stroock-Varadhan Support Theorem will be applied in the form of the following corollary.

Corollary 2.18 *Fix some $x \in U$ and $\beta, \varepsilon > 0$. In addition, let $\delta(x) > \varepsilon$ and choose some $\alpha > 0$ such that $\delta(y) > \varepsilon$ for all $y \in B(x,\alpha)$. Then $\check{P}^x(\check{\tau}_{B(x,\alpha)} > \beta) > 0$.*

Proof Put $\check{\Omega}^\alpha := \{\omega \in \check{\Omega} : \check{\tau}_{B(x,\alpha)}$ is continuous in $\omega\}$ and define $\check{\sigma}_{B(x,\alpha)} : \check{\Omega}^\alpha \to \bar{\mathbb{R}}_0^+$ by $\check{\sigma}_{B(x,\alpha)} = \check{\tau}_{B(x,\alpha)}|_{\check{\Omega}^\alpha}$. Since $\check{P}^x(\check{\Omega} \setminus \check{\Omega}^\alpha) = 0$, we deduce that $\check{Q}^x := \check{P}^x|_{\mathscr{B}(\check{\Omega}^\alpha)}$ is a probability measure on $(\check{\Omega}^\alpha, \mathscr{B}(\check{\Omega}^\alpha))$, where $\mathscr{B}(\check{\Omega}^\alpha)$ denotes the Borel σ–algebra on $\check{\Omega}^\alpha$. According to Theorem 2.17 we have that

$$\check{\Omega}^\alpha_x := \{\omega \in \check{\Omega}^\alpha : \check{X}_0(\omega) = x\} = \operatorname{supp}(\check{Q}^x) = \{\omega \in \check{\Omega}^\alpha : \check{Q}^x(N_\omega) > 0 \,\forall\, N_\omega \in \mathcal{N}_\omega\},$$

where $\mathcal{N}_\omega$ denotes the set of all open neighbourhoods of $\omega \in \check{\Omega}^\alpha$. Let $\omega' \in \check{\Omega}^\alpha_x$ be such that $\check{\sigma}_{B(x,\alpha)}(\omega') > \beta$. Then $\{\omega \in \check{\Omega}^\alpha : \check{\sigma}_{B(x,\alpha)}(\omega) > \beta\} \in \mathcal{N}_{\omega'}$, and because $\omega' \in \operatorname{supp}(\check{Q}^x)$ we deduce that $\check{Q}^x(\check{\sigma}_{B(x,\alpha)} > \beta) > 0$. Therefore, $\check{P}^x(\check{\tau}_{B(x,\alpha)} > \beta) > 0$. $\square$

Now we are in a position to prove the main result of this chapter.

Theorem 2.19 *Presume that $(a_{ij})_{i,j=1,\ldots,d} \in C(U, \mathcal{M}_d)$ is elliptic and $(b_i)_{i=1,\ldots,d} : U \to \mathbb{R}^d$ is measurable and locally bounded. Then for every $\nu \in \mathcal{P}(U)$ the martingale problem for (T,ν) on U has a solution P^ν which is unique.*

Proof Fix some arbitrary $\nu \in \mathcal{P}(U)$. Basically, our approach is to apply Theorem 2.3 and to show that the restriction to Ω of the unique probability measure $P^\nu \in P(\tilde{\Omega})$, obtained by Theorem 2.3, is indeed a uniquely defined solution to the martingale problem for (T,ν) on U. We will achieve this by showing that $P^\nu(A) = P_n^{\check{\nu}}(\check{A})$ holds true for all $n \in \mathbb{N}$ and every $A \in \tilde{\mathscr{F}}_{\tilde{\tau}_n}$. Then we will show that $\operatorname{supp}(P^\nu) \subseteq \Omega$, which by means of Remark 2.14 proves the assertion.

Part I

With the denotations of Theorem 2.3, put $(G, \mathscr{G}) := (\tilde{\Omega}, \mathscr{F}^{\tilde{X}})$ and $(\mathscr{G}_n)_{n\in\mathbb{N}} := (\tilde{\mathscr{F}}_{\tilde{\tau}_n})_{n\in\mathbb{N}}$. In addition, let $\mathscr{G}_0 := \tilde{\mathscr{F}}_0$. Note that this choice of $(G,\mathscr{G})$ and $(\mathscr{G}_n)_{n\in\mathbb{N}}$ is indeed in accordance with Theorem 2.3, since

$$\mathscr{G} = \mathscr{F}^{\tilde{X}} = \tilde{\mathscr{F}}_{\tilde{\tau}_U} = \tilde{\mathscr{F}}_0 \vee \bigcup_{n\in\mathbb{N}} \tilde{\mathscr{F}}_{\tilde{\tau}_n} = \bigvee_{n\in\mathbb{N}_0} \mathscr{G}_n$$

holds true by Remark 2.12. Let us adopt $\tilde{\tau}_0 :\equiv 0$. Then we deduce that the functions $\varphi_n : \tilde{\Omega} \to \mathfrak{P}(\tilde{\Omega})$, which appear in Theorem 2.3, are given by

$$\forall\, n \in \mathbb{N}_0 \,\forall\, \omega \in \tilde{\Omega} : \varphi_n(\omega) = \{\eta \in \tilde{\Omega} : \eta(t) = \omega(t) \,\forall\, t \in [0, \tilde{\tau}_n(\omega)]\}. \tag{2.9}$$

Moreover, we have that $\bigcap_{n\in\mathbb{N}_0} \varphi_n(\omega_n) \neq \emptyset$ holds true for any sequence $(\omega_n)_{n\in\mathbb{N}_0}$ in $\tilde{\Omega}$ satisfying $\bigcap_{n=0}^{N} \varphi_n(\omega_n) \neq \emptyset$ for all $N \in \mathbb{N}$ (cf. Remark 2.20 on p. 36). For the time being, fix an arbitrary $n \in \mathbb{N}$ and define

$$\begin{aligned}\mathcal{A}_n \;:=\; & \{\tilde{\tau}_{n-1} = \infty\} \cup \Big(\Big\{\{\tilde{X}_{(\tilde{\tau}_{n-1}+t_1)\wedge\tilde{\tau}_n} \in B_1, \ldots, \tilde{X}_{(\tilde{\tau}_{n-1}+t_m)\wedge\tilde{\tau}_n} \in B_m\} : \\ & m \in \mathbb{N}, (t_1, \ldots, t_m) \in \mathcal{H}(\mathbb{R}_0^+), B_j \in \mathscr{B}(U), j = 1, \ldots, m\Big\} \cap \{\tilde{\tau}_{n-1} < \infty\}\Big).\end{aligned}$$

By means of Lemma 2.11 we have $\tilde{\mathscr{F}}_{\tilde{\tau}_n} = \sigma(\{F \cap A : F \in \tilde{\mathscr{F}}_{\tilde{\tau}_{n-1}}, A \in \mathcal{A}_n\})$. For any $\check{\tau} \in \mathcal{S}(\mathscr{F}^{\check{X}})$ let $\check{\theta}_{\check{\tau}} : \check{\Omega} \to \check{\Omega}$ be a shift operator as in Definition 1.9. For every $\omega \in \tilde{\Omega}$ we define a probability measure ν_n^ω on $\tilde{\mathscr{F}}_{\tilde{\tau}_n}$ by

$$\nu_n^\omega(F \cap A) = \begin{cases} 0, & F \cap \varphi_{n-1}(\omega) = \emptyset \\ P_n^{\tilde{X}_{\tilde{\tau}_{n-1}}(\omega)}(\eta \in \check{\Omega} : \check{\theta}_{\check{\tau}_{n-1}}\eta \in \check{A}), & F \cap \varphi_{n-1}(\omega) \neq \emptyset, \tilde{\tau}_{n-1}(\omega) < \infty \\ 1, & \omega \in F, \tilde{\tau}_{n-1}(\omega) = \infty \end{cases}$$

for all $F \in \tilde{\mathscr{F}}_{\tilde{\tau}_{n-1}}$ and $A \in \mathcal{A}_n$. Observe that by means of Lemma B.12 the measure ν_n^ω is uniquely defined, since $\{F \cap A : F \in \tilde{\mathscr{F}}_{\tilde{\tau}_{n-1}}, A \in \mathcal{A}_n\}$ is closed under the formation of finite intersections. Let us define a Markov kernel μ_n from $(\tilde{\Omega}, \tilde{\mathscr{F}}_{\tilde{\tau}_{n-1}})$ to $(\tilde{\Omega}, \tilde{\mathscr{F}}_{\tilde{\tau}_n})$ by

$$\forall\, \omega \in \tilde{\Omega} \,\forall\, A \in \tilde{\mathscr{F}}_{\tilde{\tau}_n} : \mu_n(\omega, A) = \nu_n^\omega(A).$$

Then we infer from (2.9) that $\mu_n(\omega, A) = 0$ for all $\omega \in \tilde{\Omega}$ and $A \in \tilde{\mathscr{F}}_{\tilde{\tau}_n}$ with $\varphi_{n-1}(\omega) \cap A = \emptyset$. In addition, consider the probability measure ν^0 on $(\tilde{\Omega}, \tilde{\mathscr{F}}_0)$ defined by $\nu^0_{\tilde{X}_0} := \nu$. Hence, the assumptions of Theorem 2.3 are satisfied, and thus we obtain from Theorem 2.3 that there exists a uniquely defined probability measure P^ν on $(\tilde{\Omega}, \tilde{\mathscr{F}})$ with $P^\nu_{\tilde{X}_0} = \nu^0_{\tilde{X}_0} = \nu$ and such that $P^\nu = \int_{\tilde{\Omega}} \mu_n(\omega, \cdot) P^\nu(\mathrm{d}\omega)$ on $\tilde{\mathscr{F}}_{\tilde{\tau}_n}$.

Part II

Our aim is to show

$$\forall\, n \in \mathbb{N} \,\forall\, A \in \tilde{\mathscr{F}}_{\tilde{\tau}_n} : P^\nu(A) = P_n^{\check{\nu}}(\check{A}), \tag{2.10}$$

which can be accomplished by induction. We have $P^\nu(\tilde{X}_0 \in B) = \nu(B) = P_n^{\check{\nu}}(\check{X}_0 \in B)$ for any $B \in \mathscr{B}(U)$, i.e., $P^\nu(A) = P_n^{\check{\nu}}(\check{A})$ for all $A \in \tilde{\mathscr{F}}_0$. Let $n \in \mathbb{N}$, and assume that $P^\nu(A) = P_{n-1}^{\check{\nu}}(\check{A})$ holds for all $A \in \tilde{\mathscr{F}}_{\tilde{\tau}_{n-1}}$. Then $P^\nu(A) = P_n^{\check{\nu}}(\check{A})$ for all $A \in \tilde{\mathscr{F}}_{\tilde{\tau}_{n-1}}$, because Theorem 2.2 yields that $P_n^{\check{\nu}} = P_{n-1}^{\check{\nu}}$ on $\mathscr{F}_{\tau_{n-1}}$. For the time being, consider some $A = \{\tilde{\tau}_{n-1} = \infty\} \cup (\{\tilde{X}_{(\tilde{\tau}_{n-1}+t_1)\wedge\tilde{\tau}_n} \in B_1, \ldots, \tilde{X}_{(\tilde{\tau}_{n-1}+t_m)\wedge\tilde{\tau}_n} \in B_m\} \cap \{\tilde{\tau}_{n-1} < \infty\}) \in \mathcal{A}_n$, and observe that

$$\begin{aligned} & \omega \in \check{A} \\ \iff\; & (\omega([\check{\tau}_{n-1}(\omega) + t_1] \wedge \check{\tau}_n(\omega)), \ldots, \omega([\check{\tau}_{n-1}(\omega) + t_n] \wedge \check{\tau}_n(\omega))) \in (B_1 \times \ldots \times B_n) \end{aligned}$$

$$
\begin{aligned}
\iff\ & (\omega([\check\tau_{n-1}(\omega)+\check\tau_{n-1}(\omega)-\check\tau_{n-1}(\omega)+t_1]\wedge[\check\tau_{n-1}(\omega)+\check\tau_n(\omega)-\check\tau_{n-1}(\omega)]),\dots,\\
& \omega([\check\tau_{n-1}(\omega)+\check\tau_{n-1}(\omega)-\check\tau_{n-1}(\omega)+t_n]\wedge[\check\tau_{n-1}(\omega)+\check\tau_n(\omega)-\check\tau_{n-1}(\omega)]))\\
& \in (B_1\times\ldots\times B_n)\\
\iff\ & \big(\check\theta_{\check\tau_{n-1}}\omega([\check\tau_{n-1}(\omega)-\check\tau_{n-1}(\omega)+t_1]\wedge[\check\tau_n(\omega)-\check\tau_{n-1}(\omega)]),\dots,\\
& \check\theta_{\check\tau_{n-1}}\omega([\check\tau_{n-1}(\omega)-\check\tau_{n-1}(\omega)+t_n]\wedge[\check\tau_n(\omega)-\check\tau_{n-1}(\omega)])\big)\in(B_1\times\ldots\times B_n)\\
\iff\ & \big(\check\theta_{\check\tau_{n-1}}\omega([\check\tau_{n-1}(\check\theta_{\check\tau_{n-1}}\omega)+t_1]\wedge\check\tau_n(\check\theta_{\check\tau_{n-1}}\omega)),\dots,\\
& \check\theta_{\check\tau_{n-1}}\omega([\check\tau_{n-1}(\check\theta_{\check\tau_{n-1}}\omega)+t_n]\wedge\check\tau_n(\check\theta_{\check\tau_{n-1}}\omega))\big)\in(B_1\times\ldots\times B_n)\\
\iff\ & \check\theta_{\check\tau_{n-1}}\omega\in\check A
\end{aligned}
$$

holds for all $\omega\in\{\check\tau_{n-1}<\infty\}$, i.e.,

$$
\check\theta^{-1}_{\check\tau_{n-1}}(\check A)=\check A \text{ on } \{\check\tau_{n-1}<\infty\}. \tag{2.11}
$$

Adopt $\check\tau_0 :\equiv 0$ and define a map $\check\varphi_{n-1}:\check\Omega\to\mathfrak{P}(\check\Omega)$ by

$$
\forall\,\omega\in\check\Omega:\check\varphi_{n-1}(\omega)=\{\eta\in\check\Omega:\eta(t)=\omega(t)\,\forall\,t\in[0,\check\tau_{n-1}(\omega)]\}.
$$

By means of the strong Markov property of $(P_n^\mu)_{\mu\in\mathcal{P}(\mathbb{R}^d)}$ (cf. Theorem 2.7) we deduce that

$$
\begin{aligned}
P^\nu(F\cap A) &= \int_{\tilde\Omega}\mu_n(\cdot,F\cap A)\,\mathrm{d}P^\nu\\
&\overset{(2.11)}{=} \int_{\{\omega\in\tilde\Omega:F\cap\varphi_{n-1}(\omega)\neq\emptyset,\,\tilde\tau_{n-1}(\omega)<\infty\}} P_n^{\tilde X_{\tilde\tau_{n-1}}}(\check A)\,\mathrm{d}P^\nu+P^\nu(\tilde\tau_{n-1}=\infty)\\
&\overset{(*)}{=} \int_{\{\omega\in\check\Omega:\check F\cap\check\varphi_{n-1}(\omega)\neq\emptyset,\,\check\tau_{n-1}(\omega)<\infty\}} P_n^{\check X_{\tau_{n-1}}}(\check A)\,\mathrm{d}P_n^{\check\nu}+P_n^{\check\nu}(\check\tau_{n-1}=\infty)\\
&= \int_{\{\omega\in\check\Omega:\check F\cap\check\varphi_{n-1}(\omega)\neq\emptyset,\,\check\tau_{n-1}(\omega)<\infty\}} P_n^{\check\nu}\left(\check\theta^{-1}_{\check\tau_{n-1}}(\check A)\,\middle|\,\check{\mathscr{F}}_{\tau_{n-1}}\right)\,\mathrm{d}P_n^{\check\nu}+P_n^{\check\nu}(\check\tau_{n-1}=\infty)\\
&\overset{(2.11)}{=} P_n^{\check\nu}(\{\omega\in\check\Omega:\check F\cap\check\varphi_{n-1}(\omega)\neq\emptyset,\,\check\tau_{n-1}(\omega)<\infty\}\cap\check A)+P_n^{\check\nu}(\check\tau_{n-1}=\infty)\\
&= P_n^{\check\nu}(\check F\cap\check A)\\
&= P_n^{\check\nu}(\Psi^{-1}(F\cap A)),
\end{aligned}
$$

for all $F\in\tilde{\mathscr{F}}_{\tilde\tau_{n-1}}$ and $A\in\mathcal{A}_n$, where we have used $P^\nu=P_n^{\check\nu}\circ\Psi^{-1}$ on $\tilde{\mathscr{F}}_{\tilde\tau_{n-1}}$ in order to get $(*)$. We infer from Lemma B.12 that (2.10) holds, since $\tilde{\mathscr{F}}_{\tilde\tau_n}=\sigma(\{F\cap A:F\in\tilde{\mathscr{F}}_{\tilde\tau_{n-1}},A\in\mathcal{A}_n\})$ and $\{F\cap A:F\in\tilde{\mathscr{F}}_{\tilde\tau_{n-1}},A\in\mathcal{A}_n\}$ is closed under the formation of finite intersections.

Part III

Now we want to show that $\operatorname{supp}(P^\nu)\subseteq\Omega$. To this end observe that

$$
\tilde\Omega\setminus\Omega=\{\tilde\tau_U<\infty\}\cap\left\{\limsup_{t\to\tilde\tau_U}\delta\circ\tilde X_t>0\right\}.
$$

Hence we have to show that $P^\nu(\{\tilde\tau_U < \infty\} \cap \{\limsup_{t\to\tilde\tau_U} \delta\circ\tilde X_t > 0\}) = 0$, which is equivalent to

$$\forall\, t_0 > 0\,\forall\,\varepsilon > 0 : P^\nu\left(\{\tilde\tau_U < t_0\} \cap \left\{\limsup_{t\to\tilde\tau_U} \delta\circ\tilde X_t > \varepsilon\right\}\right) = 0.$$

For our further proceeding we fix some $\varepsilon > 0$, and additionally we choose some $0 < \rho < \varepsilon$. Let us define a sequence $(\tilde\sigma_n)_{n\in\mathbb{N}}$ of $\mathscr{F}^{\tilde X}$–stopping times by

$$\begin{aligned}\tilde\sigma_{2n} &:= \inf\{t \geq \tilde\sigma_{2n-1} : \delta\circ\tilde X_t \leq \varepsilon - \rho\}\\ \tilde\sigma_{2n+1} &:= \inf\{\tilde\sigma_{2n} \leq t < \tilde\tau_U : \delta\circ\tilde X_t \geq \varepsilon\}\end{aligned}$$

for all $n \in \mathbb{N}$, where $\tilde\sigma_1 := \inf\{0 \leq t < \tilde\tau_U : \delta\circ\tilde X_t \geq \varepsilon\}$. We also consider a sequence $(\check\sigma_n)_{n\in\mathbb{N}}$ of $\mathscr{F}^{\check X}$–stopping times, which is defined analogously with $\tilde X_t$ and $\tilde\tau_U$ replaced by $\check X_t$ and $\check\tau_U$. Observe that we have $\{\tilde\tau_U < t_0\} \cap \{\limsup_{t\to\tilde\tau_U} \delta\circ\tilde X_t > \varepsilon\} \subseteq \bigcap_{i\in\mathbb{N}}\{\tilde\sigma_i \leq t_0\}$ for every $t_0 > 0$. Therefore, it remains to show that $P^\nu\left(\bigcap_{i=1}^\infty\{\tilde\sigma_i \leq t_0\}\right) = 0$ for all $t_0 > 0$. To this end we choose some $n_0 \in \mathbb{N}$ with $U_{\varepsilon,\rho} := \{x \in U : \delta(x) > \varepsilon - \rho\} \subseteq U_{n_0}$. Note that $\tilde\tau_{U_{\varepsilon,\rho}} = \inf\{t \in \mathbb{R}_0^+ : \delta\circ\tilde X_t \leq \varepsilon - \rho\}$ and $\check\tau_{U_{\varepsilon,\rho}} = \inf\{t \in \mathbb{R}_0^+ : \delta\circ\check X_t \leq \varepsilon - \rho\}$. For any $x \in B_\varepsilon := \{y \in U : \delta(y) \geq \varepsilon\}$ choose some $\alpha_x > 0$ such that $B(x,\alpha_x) \subseteq U_{\varepsilon,\rho}$. Then we obtain that

$$\begin{aligned} & P^\nu\left(\bigcap_{i=1}^{2n}\{\tilde\sigma_i \leq t_0\}\right)\\ =\;& E_\nu\left(\prod_{i=1}^{2n}\mathbb{1}_{\{\tilde\sigma_i\leq t_0\}}\right)\\ =\;& E_\nu\left(E_\nu\left(\mathbb{1}_{\{\tilde\sigma_{2n}\leq t_0\}}\prod_{i=1}^{2n-1}\mathbb{1}_{\{\tilde\sigma_i\leq t_0\}}\,\middle|\,\tilde{\mathscr{F}}_{\tilde\sigma_{2n-1}}\right)\right)\\ \overset{(*)}{=}\;& E_\nu\left(\prod_{i=1}^{2n-1}\mathbb{1}_{\{\tilde\sigma_i\leq t_0\}}E_\nu\left(\mathbb{1}_{\{\tilde\sigma_{2n}\leq t_0\}}\,\middle|\,\tilde{\mathscr{F}}_{\tilde\sigma_{2n-1}}\right)\right)\\ \leq\;& E_\nu\left(\prod_{i=1}^{2n-1}\mathbb{1}_{\{\tilde\sigma_i\leq t_0\}}P^\nu\left(\inf\{t\geq 0 : \delta\circ\tilde X_{t+\tilde\sigma_{2n-1}} \leq \varepsilon-\rho\} \leq t_0\,\middle|\,\tilde{\mathscr{F}}_{\tilde\sigma_{2n-1}}\right)\right)\\ \overset{(**)}{\leq}\;& E_\nu\left(\prod_{i=1}^{2n-1}\mathbb{1}_{\{\tilde\sigma_i\leq t_0\}}P^{\tilde X_{\tilde\sigma_{2n-1}}}(\tilde\tau_{U_{\varepsilon,\rho}} \leq t_0)\right)\\ \overset{(***)}{=}\;& E_\nu\left(\prod_{i=1}^{2n-1}\mathbb{1}_{\{\tilde\sigma_i\leq t_0\}}P_{n_0}^{\tilde X_{\tilde\sigma_{2n-1}}}(\check\tau_{U_{\varepsilon,\rho}} \leq t_0)\right)\\ \overset{(****)}{\leq}\;& E_\nu\left(\prod_{i=1}^{2n-1}\mathbb{1}_{\{\tilde\sigma_i\leq t_0\}}P_{n_0}^{\tilde X_{\tilde\sigma_{2n-1}}}\left(\check\tau_{B\left(\tilde X_{\tilde\sigma_{2n-1}},\alpha_{\tilde X_{\tilde\sigma_{2n-1}}}\right)} \leq t_0\right)\right) \end{aligned} \tag{2.12}$$

holds true for all $n \in \mathbb{N}$ and every $t_0 > 0$.

To

$(*)$: Note that $\tilde{\sigma}_m$ is $\tilde{\mathscr{F}}_{\tilde{\sigma}_{2n-1}}$–$\mathscr{B}(\bar{\mathbb{R}}_0^+)$–measurable for all $m \leq 2n-1$.

$(**)$: By Theorem 2.7 we have that $(P_n^\mu)_{\mu\in\mathcal{P}(\mathbb{R}^d)}$ satisfies the strong Markov property for any $n \in \mathbb{N}$. Let $\tilde{\theta}_{\tilde{\sigma}_{2n-1}} : \tilde{\Omega} \to \tilde{\Omega}$ be a shift operator as in Definition 1.9. Then we obtain

$$\begin{aligned}
& P^\nu(\inf\{t \geq 0 : \delta \circ \tilde{X}_{t+\tilde{\sigma}_{2n-1}} \leq \varepsilon - \rho\} \leq t_0 | \tilde{\mathscr{F}}_{\tilde{\sigma}_{2n-1}}) \\
= & \sum_{k\in\mathbb{N}} P^\nu((\tilde{\theta}^{-1}_{\tilde{\sigma}_{2n-1}}\{\tilde{\tau}_{U_{\varepsilon,\rho}} \leq t_0\}) \cap \{\tilde{\tau}_{k-1} \leq \tilde{\sigma}_{2n} < \tilde{\tau}_k\} | \tilde{\mathscr{F}}_{\tilde{\sigma}_{2n-1}}) \\
= & \sum_{k\in\mathbb{N}} P_k^{\check{\nu}}((\check{\theta}^{-1}_{\check{\sigma}_{2n-1}}\{\check{\tau}_{U_{\varepsilon,\rho}} \leq t_0\}) \cap \{\check{\tau}_{k-1} \leq \check{\sigma}_{2n} < \check{\tau}_k\} | \check{\mathscr{F}}_{\check{\sigma}_{2n-1}}) \\
\leq & \sum_{k\in\mathbb{N}} P_k^{\check{X}_{\check{\sigma}_{2n-1}}}(\{\check{\tau}_{U_{\varepsilon,\rho}} \leq t_0\} \cap \check{\theta}_{\check{\sigma}_{2n-1}}\{\check{\tau}_{k-1} \leq \check{\sigma}_{2n} < \check{\tau}_k\}) \\
= & \sum_{k\in\mathbb{N}} P^{\tilde{X}_{\tilde{\sigma}_{2n-1}}}(\{\tilde{\tau}_{U_{\varepsilon,\rho}} \leq t_0\} \cap \tilde{\theta}_{\tilde{\sigma}_{2n-1}}\{\tilde{\tau}_{k-1} \leq \tilde{\sigma}_{2n} \wedge \tilde{\tau}_U < \tilde{\tau}_k\}) \\
= & P^{\tilde{X}_{\tilde{\sigma}_{2n-1}}}(\tilde{\tau}_{U_{\varepsilon,\rho}} \leq t_0)
\end{aligned}$$

P^ν–a.s. on $\{\tilde{\sigma}_{2n-1} < \infty\}$, because $P^\nu(A) = P_n^{\check{\nu}}(\check{A})$ for all $n \in \mathbb{N}$ and every $A \in \tilde{\mathscr{F}}_{\tilde{\tau}_n}$.

$(***)$: Since $U_{\varepsilon,\rho} \subseteq U_{n_0}$, this follows from $P^{\tilde{X}_{\tilde{\sigma}_{2n-1}}}(A) = P_{n_0}^{\tilde{X}_{\tilde{\sigma}_{2n-1}}}(\check{A})$ for all $A \in \tilde{\mathscr{F}}_{\tilde{\tau}_{n_0}}$.

$(****)$: Recall that $(P_{n_0}^x \circ \check{X}_0) = \delta_x$ for every $x \in \mathbb{R}^d$, where δ_x denotes the Dirac measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Thus we infer that $\check{X}_0 \in B_\varepsilon$ $P_{n_0}^{\tilde{X}_{\tilde{\sigma}_{2n-1}}(\omega)}$–a.s. for all $\omega \in \tilde{\Omega}$ with $\sigma_{2n-2} < \infty$. We obtain $(****)$, because $B(x, \alpha_x) \subseteq U_{\varepsilon,\rho}$ for any $x \in B_\varepsilon$.

By Remark 2.16 we have that $\check{\tau}_{B(x,\alpha_x)}$ is $P_{n_0}^x$–a.s. continuous for each $x \in U$. Moreover, Theorem 2.9 yields that $P_{n_0}^x \circ \check{X}_t$ has a λ_d–density for every $x \in U$ and each $t \in \mathbb{R}_0^+$, i.e., in particular, $P_{n_0}^x(\check{\tau}_{B(x,\alpha_x)} = t_0) \leq P_{n_0}^x(\check{X}_{t_0} \in \partial B(x,\alpha_x)) = 0$ for any $t_0 > 0$ and all $x \in B_\varepsilon$. Therefore, $\mathbb{1}_{\{\check{\tau}_{B(x,\alpha_x)} \leq t_0\}}$ is $P_{n_0}^x$–a.s. continuous for all $x \in B_\varepsilon$, which by means of Theorem 2.8 in conjunction with Theorem 2.7 yields that $P_{n_0}^{(\cdot)}(\check{\tau}_{B(x,\alpha_x)} \leq t_0)$ is continuous on B_ε. In addition, we infer from Corollary 2.18 that $\gamma := \max_{x\in B_\varepsilon} P_{n_0}^x(\check{\tau}_{B(x,\alpha_x)} \leq t_0) < 1$, since otherwise the continuity of $P_{n_0}^{(\cdot)}(\check{\tau}_{B(x,\alpha_x)} \leq t_0)$ on B_ε would yield the existence of some $y \in B_\varepsilon$ with $P_{n_0}^y(\check{\tau}_{B(x,\alpha_x)} \leq t_0) = 1$, which were a contradiction to Corollary 2.18. By our considerations above we do now deduce from $\delta \circ \tilde{X}_{\tilde{\sigma}_{2n-1}} \geq \varepsilon$ that

$$P^\nu\left(\bigcap_{i=1}^{2n}\{\tilde{\sigma}_i \leq t_0\}\right) \overset{(2.12)}{\leq} \gamma E_\nu\left(\prod_{i=1}^{2n-1}\mathbb{1}_{\{\tilde{\sigma}_i \leq t_0\}}\right) = \gamma P^\nu\left(\bigcap_{i=1}^{2n-1}\{\tilde{\sigma}_i \leq t_0\}\right) \leq \gamma P^\nu\left(\bigcap_{i=1}^{2(n-1)}\{\tilde{\sigma}_i \leq t_0\}\right)$$

holds true for all $n \in \mathbb{N}$ and any $t_0 > 0$. By continuing this process we obtain that

$$\forall\, n \in \mathbb{N}\, \forall\, t_0 > 0 : P^\nu\left(\bigcap_{i=1}^{2n}\{\tilde{\sigma}_i \leq t_0\}\right) \leq \gamma^{n-1} P^\nu\left(\{\tilde{\sigma}_1 \leq t_0\} \cap \{\tilde{\sigma}_2 \leq t_0\}\right) \leq \gamma^{n-1},$$

which results in

$$\forall\, t_0 > 0 : P^\nu\left(\bigcap_{i=1}^{\infty}\{\tilde{\sigma}_i \leq t_0\}\right) \leq \lim_{n\to\infty} \gamma^{n-1} = 0.$$

As mentioned above this shows that $P^\nu|_\Omega \in \mathcal{P}(\Omega)$. Now we infer from Remark 2.14 that $P^\nu|_\Omega$ is the unique solution to the martingale problem for (T,ν) on U. $\square$

Remark 2.20 Note that the property " $\bigcap_{n\in\mathbb{N}_0} \varphi_n(\omega_n) \neq \emptyset$ holds for any sequence $(\omega_n)_{n\in\mathbb{N}}$ in G satisfying $\bigcap_{n=0}^N \varphi_n(\omega_n) \neq \emptyset$ for all $N \in \mathbb{N}$", which we have used in Part I of the proof of Theorem 2.19, does hold if $(G,\mathscr{G}) = (\tilde{\Omega}, \mathscr{F}^{\tilde{X}})$, but not if $(G,\mathscr{G}) = (\Omega, \mathscr{F}^X)$.

To begin with, we show that this property holds if $(G,\mathscr{G}) = (\tilde{\Omega}, \mathscr{F}^{\tilde{X}})$. To this end, consider a sequence $(\omega_n)_{n\in\mathbb{N}_0}$ in $\tilde{\Omega}$ such that $\bigcap_{n=0}^N \varphi_n(\omega_n) \neq \emptyset$ for all $N \in \mathbb{N}$. Since $\varphi_n(\omega_n) \cap \varphi_{n+1}(\omega_{n+1}) \neq \emptyset$, we infer that $\omega_{n+1} = \omega_n$ on $[0, \tilde{\tau}_n(\omega_n)]$ for all $n \in \mathbb{N}$. Thus, $\tilde{\tau}_n(\omega_n) = \tilde{\tau}_n(\omega_{n+1}) \leq \tilde{\tau}_{n+1}(\omega_{n+1})$ for all $n \in \mathbb{N}$, i.e., $\lim_{n\to\infty} \tau_n(\omega_n)$ exists. Then define $\omega \in \tilde{\Omega}$ by $\omega = \omega_n$ on $[0, \tau_n(\omega_n)]$ for all $n \in \mathbb{N}$ and $\omega \equiv \Delta$ on $[\lim_{n\to\infty} \tau_n(\omega_n), \infty)$. Note that ω is indeed an element of $\tilde{\Omega}$. Moreover, we have $\omega \in \bigcap_{n\in\mathbb{N}_0} \varphi_n(\omega_n)$.

That the aforementioned property does not hold if $(G,\mathscr{G}) = (\Omega, \mathscr{F}^X)$ can be shown by constructing a sequence $(\omega_n)_{n\in\mathbb{N}_0}$ in Ω as follows: Assume that $d = 1$, $U = (-1,1)$ and $U_n = (n^{-1} - 1, 1 - n^{-1})$ for all $n \in \mathbb{N}$. Furthermore, let $\omega \in \tilde{\Omega}$ be given by $\omega(t) = t\sin((1-t)^{-1})\mathbb{1}_{\{t<1\}} + \Delta\mathbb{1}_{\{t\geq 1\}}$, consider $\omega_0 :\equiv 0 \in \Omega$, and for any $n \in \mathbb{N}_0$ define $\omega_n \in \Omega$ by

$$\omega_n(t) = t\sin((1-t)^{-1})\mathbb{1}_{\{t<\tilde{\tau}_n(\omega)\}} + \tilde{\tau}_n(\omega)\sin((1-\tilde{\tau}_n(\omega))^{-1})\mathbb{1}_{\{t\geq\tilde{\tau}_n(\omega)\}}$$

for all $t \in \mathbb{R}_0^+$. Then $(\omega_n)_{n\in\mathbb{N}_0}$ is a sequence in Ω which satisfies $\bigcap_{n=0}^N \varphi_n(\omega_n) \neq \emptyset$ for all $N \in \mathbb{N}$. Any element in $\bigcap_{n\in\mathbb{N}_0} \varphi_n(\omega_n)$ has to coincide with ω on $[0,1)$, but there does not exist any element in Ω which coincides with ω on $[0,1)$, since ω does not have a left-hand limit as $t \uparrow 1$. Therefore, $\bigcap_{n\in\mathbb{N}_0} \varphi_n(\omega_n) = \emptyset$.

The conclusion that the aforementioned property holds if $(G,\mathscr{G}) = (\tilde{\Omega}, \mathscr{F}^{\tilde{X}})$, but not if $(G,\mathscr{G}) = (\Omega, \mathscr{F}^X)$, is the reason why we consider $\tilde{\Omega}$ at all. $\Diamond$

Now that we have given sufficient conditions under which the martingale problem for T on U has a unique solution, we are goint to present a few properties of this solution.

Lemma 2.21 *Assume that $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ satisfy the assumptions of Theorem 2.19. Furthermore, let $(P^\nu)_{\nu\in\mathcal{P}(U)}$ be the unique solution to the martingale problem for T on U. Then $P^{(\cdot)}(A) : U \to [0,1]$ is $\mathscr{B}(U)$–$\mathscr{B}([0,1])$–measurable for every $A \in \Omega$.*

Proof The assertion can be proven on the lines of the ideas in 6.7.4 in [SV79]. $\square$

Theorem 2.22 *Assume that $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ are as in Theorem 2.19. Furthermore, let $(P^\nu)_{\nu\in\mathcal{P}(U)}$ denote the unique solution to the martingale problem for T on U. Then a family $\mathscr{P} \subseteq (P^\nu)_{\nu\in\mathcal{P}(U)}$ is relatively weakly compact if*

$$\lim_{k\to\infty} \sup_{P\in\mathscr{P}} P(\|X_0\|_2 \geq k) = 0. \tag{2.13}$$

Proof The assertion can be proven along the lines of the proof of Theorem 2.5. $\square$

Theorem 2.23 *Presume that $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ satisfy the assumptions of Theorem 2.19, and let $(P^\nu)_{\nu\in\mathcal{P}(U)}$ be the unique solution to the martingale problem for T on U. If $(b_i)_{i=1,\dots,d}$ is continuous, then $(P^\nu)_{\nu\in\mathcal{P}(U)}$ satisfies the Feller property.*

Proof *Definition 1.12 (i):* Let $x \in U$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in U such that $x_n \to x$ as $n \to \infty$. We want to apply Theorem 2.22 in order to deduce that $(P^{x_n})_{n\in\mathbb{N}}$ is relatively weakly compact. To this end we have to show that (2.13) is satisfied. For this purpose choose some $\varepsilon > 0$. Since $x_n \to x$, there exists an $n_0 \in \mathbb{N}$ such that $\|x - x_n\|_2 \le \varepsilon$ for all $n \ge n_0$. With $k > \max(\max\{\|x_n\|_2 : n < n_0\}, \|x\|_2 + \varepsilon)$ we obtain that $\sup_{n\in\mathbb{N}} P^{x_n}(\|X_0\|_2 \ge k) = 0$, which shows that (2.13) is satisfied. Consequently, by means of Theorem 2.22, the sequence $(P^{x_n})_{n\in\mathbb{N}}$ is relatively compact in the weak topology. In particular, we deduce that $\underline{(P^{x_n})_{n\in\mathbb{N}}} \subseteq (P^x)_{x\in U}$ has a subsequence $(P^{x_{n_k}})_{k\in\mathbb{N}}$ converging weakly to some $\omega\text{–}\lim_{k\to\infty} P^{x_{n_k}} \in \overline{(P^x)}_{x\in U}$. Note that by Remark 13.13 in [Kle06] the weak limit $\omega\text{–}\lim_{k\to\infty} P^{x_{n_k}}$ is uniquely defined. The idea is to show that $Q := \omega\text{–}\lim_{k\to\infty} P^{x_{n_k}} \in \mathcal{P}(\Omega)$ is a solution to the martingale problem for (T, δ_x) on U. Then, by the uniqueness of the solution, this yields that $\omega\text{–}\lim_{k\to\infty} P^{x_{n_k}} = P^x$. We deduce from 7.8 Theorem (a) in Chapter 3 in [EK86] that $P^{x_{n_k}}_{(X_{t_1},\dots,X_{t_n})} \to Q_{(X_{t_1},\dots,X_{t_n})}$ weakly as $k \to \infty$ for all $n \in \mathbb{N}, t_1, \dots, t_n \in \mathbb{R}_0^+$. Thus it remains to show that Q satisfies Definition 2.10 (i) for δ_x and Definition 2.10 (ii).

Definition 2.10 (i): Consider the sequence $(\varphi_m)_{m\in\mathbb{N}}$ of functions $\varphi_m \in C_b(U)$ with $\varphi_m \equiv 1$ on $B[x, 1/m]$, $\varphi_m \equiv 0$ on $U \setminus B[x, 2/m]$ and such that $\varphi_m(y) = (2/m - \|y - x\|_2)m$ for all $y \in B[x, 2/m] \setminus B[x, 1/m]$. Fix some arbitrary $m \in \mathbb{N}$ and choose a $k_m \in \mathbb{N}$ with $x_{n_k} \in B[x, 1/m]$ for any $k \ge k_m$. Note that such a k_m exists, since $x_n \to x$ as $n \to \infty$. Then $E_{x_{n_k}}(\varphi_m \circ X_0) = 1$ for all $k \ge k_m$, which results in $E_Q(\varphi_m \circ X_0) = \lim_{k\to\infty} E_{x_{n_k}}(\varphi_m \circ X_0) = 1$, where E_Q denotes the expectation with respect to Q. Thus, since $m \in \mathbb{N}$ was chosen arbitrarily and because X_0 is continuous, we infer that

$$Q_{X_0}(\{x\}) = E_Q\left(\lim_{m\to\infty} \varphi_m \circ X_0\right) \overset{\text{DCT}}{=} \lim_{m\to\infty} E_Q\left(\varphi_m \circ X_0\right) = 1,$$

which shows that Q satisfies Definition 2.10 (i) for δ_x.

Definition 2.10 (ii): By the $\mathscr{F}^X$–martingale property of $(M_t^{n,f})_{t\in\mathbb{R}_0^+}$ under $P^{x_{n_k}}$ we have that

$$\int_A M_t^{n,f} \, \mathrm{d}P^{x_{n_k}} = \int_A M_s^{n,f} \, \mathrm{d}P^{x_{n_k}}$$

holds true for any $f \in C_K^2(U)$, each $k \in \mathbb{N}$, all $s, t \in \mathbb{R}_0^+$ with $s \le t$ and every $A \in \mathscr{F}_s$.

Let $t \in \mathbb{R}_0^+$ and let $A \in \mathscr{F}_t$ be closed. In view of Lemma B.21 there exists a sequence $(g_m)_{m\in\mathbb{N}} \subseteq C(\Omega, [0,1])$ with $g_m \downarrow \mathbb{1}_A$ pointwise as $m \to \infty$. Hence we obtain that

$$\begin{aligned}
\lim_{k\to\infty} \int_A f \circ X_{t\wedge\tau_n} \, \mathrm{d}P^{x_{n_k}} &\overset{\text{DCT}}{=} \lim_{k\to\infty} \lim_{m\to\infty} \int_\Omega g_n(f \circ X_{t\wedge\tau_n}) \, \mathrm{d}P^{x_{n_k}} \\
&= \lim_{m\to\infty} \lim_{k\to\infty} \int_\Omega g_n(f \circ X_{t\wedge\tau_n}) \, \mathrm{d}P^{x_{n_k}} \\
&= \lim_{m\to\infty} \int_\Omega g_n(f \circ X_{t\wedge\tau_n}) \, \mathrm{d}Q \\
&\overset{\text{DCT}}{=} \int_A f \circ X_{t\wedge\tau_n} \, \mathrm{d}Q
\end{aligned}$$

and similarly

$$\lim_{k\to\infty} \int_A \int_{[0,t\wedge\tau_n]} Tf \circ X_{u\wedge\tau_n} \lambda(\mathrm{d}u) \, \mathrm{d}P^{x_{n_k}} \overset{\text{DCT}}{=} \int_{[0,t\wedge\tau_n]} \lim_{k\to\infty} \int_A Tf \circ X_{u\wedge\tau_n} \, \mathrm{d}P^{x_{n_k}} \lambda(\mathrm{d}u)$$

$$= \int_{[0,t\wedge\tau_n]}\int_A Tf\circ X_{u\wedge\tau_n}\,\mathrm{d}Q\lambda(\mathrm{d}u)$$
$$= \int_A\int_{[0,t\wedge\tau_n]} Tf\circ X_{u\wedge\tau_n}\lambda(\mathrm{d}u)\,\mathrm{d}Q$$

hold for all $f \in C_K^2(U)$, since $Tf \in C_b(U)$, and because X_σ is continuous on Ω for each $\sigma \in \mathcal{S}_b(\mathscr{F}^X)$. Thus,

$$\int_A M_t^{n,f}\,\mathrm{d}Q = \lim_{k\to\infty}\int_A M_t^{n,f}\,\mathrm{d}P^{x_{n_k}} = \lim_{k\to\infty}\int_A M_s^{n,f}\,\mathrm{d}P^{x_{n_k}} = \int_A M_s^{n,f}\,\mathrm{d}Q$$

holds for any $f \in C_K^2(U)$, each $k\in\mathbb{N}$, all $s,t\in\mathbb{R}_0^+$ with $s\le t$ and every closed $A\in\mathscr{F}_s$. Since $\int_{(\cdot)} M_t^{n,f}\,\mathrm{d}Q$ and $\int_{(\cdot)} M_s^{n,f}\,\mathrm{d}Q$ are finite measures on $(\Omega,\mathscr{F}_s)$ with $\int_\Omega M_t^{n,f}\,\mathrm{d}Q = \int_\Omega M_s^{n,f}\,\mathrm{d}Q$, and because $\mathscr{F}_s$ is generated by $\{A\in\mathscr{F}_s : A \text{ is closed}\}$, which is closed under the formation of finite intersections, we conclude that $\int_A M_t^{n,f}\,\mathrm{d}Q = \int_A M_s^{n,f}\,\mathrm{d}Q$ for all $A\in\mathscr{F}_s$, which shows that Q satisfies Definition 2.10 (ii). Therefore, $(P^\nu)_{\nu\in\mathcal{P}(U)}$ satisfies Definition 1.12 (i).

Definition 1.12 (ii): Fix an arbitrary $x\in U$ and $f\in C_b(U)$. By means of the Dominated Convergence Theorem and the continuity of $X_{(\cdot)}(\omega)$ on $\mathbb{R}_0^+$ for any $\omega\in\Omega$, we have that

$$\lim_{t\to 0} E_x(f\circ X_t) \overset{\text{DCT}}{=} E_x(f\circ \lim_{t\to 0} X_t) = E_x\left(f\circ X_0\right),$$

and hence we deduce that $P_{X_t}^x \xrightarrow{\text{w}} P_{X_0}^x$ as $t\to 0$, which is equivalent to $X_t \xrightarrow{P^x} X_0$ as $t\to\infty$, since $P_{X_0}^x = \delta_x$. Therefore, $(P^\nu)_{\nu\in\mathcal{P}(U)}$ satisfies Definition 1.12 (ii), and thus we conclude that $(P^\nu)_{\nu\in\mathcal{P}(U)}$ satisfies the Feller property. □

According to Theorem 1.1 in Chapter 2 in [Pin95] we have that $\sup_{x\in D} E_x(\tau_D)\in\mathbb{R}_0^+$ for every domain $D\subseteq U$ with $\bar{D}\subseteq U$. In particular, this yields that $\tau_n\in\mathscr{L}^1(\Omega,P^\nu)$ and thus $\tau_n<\infty$ P^ν–a.s. for all $n\in\mathbb{N}$ and any $\nu\in\mathcal{P}(U)$.

Theorem 2.24 *Presume that $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ are as in Theorem 2.19. Then $(P^\nu)_{\nu\in\mathcal{P}(U)}$, the unique solution to the martingale problem for T on U, satisfies the strong Markov property.*

Proof Fix some arbitrary $\nu\in\mathcal{P}(U)$, $\tau\in\mathcal{S}_f(P^\nu,\mathscr{F}^X)$. The approach to prove the assertion is quite similar to the idea which we have employed in the proof of Lemma 2.23 in order to prove that Definition 1.12 (i) is satified. For any $\sigma\in\mathcal{S}(\mathscr{F}^X)$ let $\theta_\sigma:\Omega\to\Omega$ be a shift operator as in Definition 1.9. In view of Theorem B.19 we consider a regular $\mathscr{F}_\tau$–conditional distribution $P^\nu_{\theta_\tau|\mathscr{F}_\tau}$ of θ_τ, i.e., $P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\cdot,A) = P^\nu(\theta_\tau^{-1}(A)|\mathscr{F}_\tau)$ P^ν–a.s. for all $A\in\mathscr{F}$, and we show that for P^ν–a.a. $\omega\in\Omega$ the probability measure $P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\omega,\cdot)\in\mathcal{P}(\Omega)$ solves the martingale problem for $(T,\delta_{X_\tau(\omega)})$ on U. Then we infer from the uniqueness of the solution that $P^\nu(\theta_\tau^{-1}(A)|\mathscr{F}_\tau) = P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\cdot,A) = P^{X_\tau}(A)$ P^ν–a.s. for all $A\in\mathscr{F}$, which proves the assertion. Since all our considerations are only P^ν–a.s., and because $\tau<\infty$ P^ν–a.s., we can assume w.l.o.g. that $\tau<\infty$. In the following we will denote by $P^\nu_{\theta_0|\mathscr{F}_\tau}$ a regular $\mathscr{F}_\tau$–conditional distribution of θ_0, i.e., $P^\nu_{\theta_0|\mathscr{F}_\tau}(\cdot,A) = P^\nu(A|\mathscr{F}_\tau)$ P^ν–a.s. for all $A\in\mathscr{F}$.

Now we show that $P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\omega,\cdot)$ satisfies Definition 2.10 (i) for $\delta_{X_\tau(\omega)}$ and Definition 2.10 (ii) for P^ν–a.a. $\omega\in\Omega$.

Definition 2.10 (i): By means of the P^ν–a.s. uniqueness of the regular conditional distributions (cf. Theorem B.19) we obtain that

$$P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\omega, \{X_0 \in \cdot\}) = P^\nu_{\theta_0|\mathscr{F}_\tau}(\omega, \{X_\tau \in \cdot\}) = \mathbb{1}_{\{X_\tau \in \cdot\}}(\omega) = \delta_{X_\tau(\omega)}$$

holds true for P^ν–a.a. $\omega \in \Omega$, since X_τ is $\mathscr{F}_\tau$–$\mathscr{B}(\hat{U})$–measurable, where $\mathscr{B}(\hat{U})$ denotes the Borel σ–algebra on $\hat{U}$. This shows that $P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\omega, \cdot)$ satisfies Definition 2.10 (i) for $\delta_{X_\tau(\omega)}$ for P^ν–a.a. $\omega \in \Omega$.

Definition 2.10 (ii): Fix some $n \in \mathbb{N}$ and $f \in C^2_K(U)$. In addition, let $s, t \in \mathbb{R}^+_0$ with $s \leq t$. Observe that $(M^{n,f}_u)_{u \in \mathbb{R}^+_0}$ is uniformly P^ν–integrable, since

$$\forall\, u \in \mathbb{R}^+_0 : |M^{n,f}_u| \leq \|f\|_\infty + \|Tf\|_\infty \tau_n$$

and $\|f\|_\infty + \|Tf\|_\infty \tau_n \in \mathscr{L}^1(\Omega, P^\nu)$, because $\tau_n \in \mathscr{L}^1(\Omega, P^\nu)$. By the $\mathscr{F}^X$–martingale property of $(M^{n,f}_u)_{u \in \mathbb{R}^+_0}$ in conjunction with the Optional Sampling Theorem (cf. Theorem B.18) we have that $\int_A M^{n,f}_{t+\tau}\, \mathrm{d}P^\nu = \int_A M^{n,f}_{s+\tau}\, \mathrm{d}P^\nu$ holds true for all $A \in \mathscr{F}_{s+\tau}$. By means of the P^ν–a.s. uniqueness of the regular conditional distributions and by means of Lemma B.20 this results in

$$\begin{aligned}
\int_A [M^{n,f}_t - M^{n,f}_s](\omega) P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\cdot, \mathrm{d}\omega) &= \int_{\theta_\tau^{-1}(A)} [M^{n,f}_t - M^{n,f}_s] \circ \theta_\tau(\omega) P^\nu_{\theta_0|\mathscr{F}_\tau}(\cdot, \mathrm{d}\omega) \\
&\overset{L.B.20}{=} E_\nu \left(\mathbb{1}_{\theta_\tau^{-1}(A)} [M^{n,f}_{t+\tau} - M^{n,f}_{s+\tau}] \middle| \mathscr{F}_\tau \right) \\
&= 0
\end{aligned}$$

P^ν–a.s. for every $A \in \mathscr{F}_s$, where $\int_A [M^{n,f}_t - M^{n,f}_s](\omega) P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\cdot, \mathrm{d}\omega) : \Omega \to \mathbb{R}$ is given by $\int_A [M^{n,f}_t - M^{n,f}_s](\omega) P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\cdot, \mathrm{d}\omega)(\eta) = \int_A [M^{n,f}_t - M^{n,f}_s](\omega) P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\eta, \mathrm{d}\omega)$ for all $\eta \in \Omega$. The function $\int_{\theta_\tau^{-1}(A)} [M^{n,f}_t - M^{n,f}_s] \circ \theta_\tau(\omega) P^\nu_{\theta_0|\mathscr{F}_\tau}(\cdot, \mathrm{d}\omega) : \Omega \to \mathbb{R}$ is given analogously. Hence, $P^\nu_{\theta_\tau|\mathscr{F}_\tau}(\omega, \cdot)$ satisfies Definition 2.10 (ii) for P^ν–a.a. $\omega \in \Omega$, which proves the assertion. □

Remark 2.25 In the proof of Theorem 2.7 we have claimed that the assertions can be proven on the lines of the proofs of Theorem 2.23 and Theorem 2.24. Well, if we want to prove that the unique solution to the martingale problem on $\mathbb{R}^d$ for some operator satisfies the strong Markov property, then it is a little bit more intricate to show that Definition 2.1 (ii) is satisfied. Hence we will briefly scetch the main idea how to show this point.

Assume that the martingale problem on $\mathbb{R}^d$ for some second–order differential operator T on $C^2_K(\mathbb{R}^d)$, as in Section 2.1, has a unique solution $(P^\nu)_{\nu \in \mathcal{P}(\mathbb{R}^d)} \subseteq \mathcal{P}(\check{\Omega})$. Let $\tau \in \mathcal{S}_f(P^\nu, \mathscr{F}^{\check{X}})$ and $f \in C^2_K(\mathbb{R}^d)$. Moreover, let $\nu \in P(\mathbb{R}^d)$, and let $(D_n)_{n \in \mathbb{N}}$ be a sequence of bounded domains $D_n \subseteq \mathbb{R}^d$ with $\mathbb{R}^d = \bigcup_{n \in \mathbb{N}} D_n$ and such that $\bar{D}_n \subsetneq D_{n+1}$ for all $n \in \mathbb{N}$. Then we can apply the Optional Sampling Theorem (cf. Theorem B.18) in order to deduce that $\left(f \circ \check{X}_{t \wedge D_n} - \int_{[0, t \wedge D_n]} Tf \circ \check{X}_s \lambda(\mathrm{d}s) \right)_{t \in \mathbb{R}^+_0}$ is an $\mathscr{F}^{\check{X}}$–martingale under P^ν. Let $\omega \in \Omega$ and consider the measure $P^\nu(\theta_\tau^{-1}(\cdot)|\mathscr{F}_\tau)(\omega) \in \mathcal{P}(\Omega)$. Along the lines of the proof of Theorem 2.24 we obtain that $\left(f \circ \check{X}_{t \wedge \check{\tau}_{D_n}} - \int_{[0, t \wedge \check{\tau}_{D_n}]} Tf \circ \check{X}_s \lambda(\mathrm{d}s) \right)_{t \in \mathbb{R}^+_0}$ is an $\mathscr{F}^{\check{X}}$–martingale under $P^\nu(\theta_\tau^{-1}(\cdot)|\mathscr{F}_\tau)(\omega)$ for any $n \in \mathbb{N}$. By considering the limits as $n \to \infty$ this results in $\left(f \circ \check{X}_t - \int_{[0,t]} Tf \circ \check{X}_s \lambda(\mathrm{d}s)\right)_{t\in\mathbb{R}_0^+}$ being an $\mathscr{F}^{\check{X}}$–martingale under $P^\nu(\theta_\tau^{-1}(\cdot)|\mathscr{F}_\tau)(\omega)$, since $\lim_{n\to\infty} \check{\tau}_{D_n} = \check{\tau}_{\mathbb{R}^d} = \infty$. This shows that Definition 2.1 (ii) is satisfied if we proceed as in the proof of Theorem 2.24 in order to show that $(P^\nu)_{\nu\in\ P(\mathbb{R}^d)}$ satisfies the strong Markov property. $\Diamond$

Theorem 2.26 *Assume that $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ satisfy the assumptions of Theorem 2.19, and let $(P^\nu)_{\nu\in\mathcal{P}(U)}$ denote the unique solution to the martingale problem for T on U. Then $\int_\Omega Y\ dP^{(\cdot)}$ is continuous on U for any $Y \in \mathcal{B}(\Omega)$ which is continuous P^y–a.s. for every $y \in U$.*

Proof The assertion can be proven on the lines of the proof of Theorem 2.8. $\square$

2.3 Diffusions

The main purpose of this thesis is to study certain properties of diffusion semigroups. Therefore, as in Chapter 1, the motivation for the theory developed so far is to construct a diffusion process in a domain $U \subseteq \mathbb{R}^d$. In this section we will give a second definition of diffusion processes, and we will briefly discuss its relation to the Feller diffusions which we have defined in the previous chapter (cf. Definition 1.30).

Throughout this section let $U \subsetneq \mathbb{R}^d$ be a domain.

Definition 2.27 *Presume that $(a_{ij})_{i,j=1,\dots,d}$, $(b_i)_{i=1,\dots,d}$ and T are as in Theorem 2.19, and let $(P^\nu)_{\nu\in\mathcal{P}(U)}$ denote the unique solution to the martingale problem for T on U. For any $\nu \in \mathcal{P}(U)$ we call (X, P^ν) the* **diffusion** *with respect to T and with initial distribution ν. Furthermore, the associated sub–Markov semigroup $(P_t)_{t\in\mathbb{R}_0^+}$ of transition kernels is referred to as* **diffusion semigroup**. *In the spirit of referring to Δ as cemetery state we say that X is killed at τ_U, and we call τ_U the* **lifetime** *of the diffusion. Note that this killing is different from the killing in Subsection 1.3.2.*

Now we want to discuss briefly the relation between the diffusions in the sense of Definition 2.27 and the Feller diffusions in U as given by Definition 1.30.

A Feller diffusion $(\acute{X}^U, \acute{P}_U^\nu)$ in U with respect to T_U and with initial distribution $\nu \in \mathcal{P}(U)$ clearly satisfies Definition 2.10 (i). Moreover, we deduce from Lemma 1.20 and the Optional Sampling Theorem (cf. Theorem B.18) that $(f \circ \acute{X}^U_{t\wedge\tau_n} - \int_{[0,t\wedge\tau_n]} (T_U f) \circ \acute{X}^U_s\ \lambda(\mathrm{d}s))_{t\in\mathbb{R}_0^+}$ is an $(\acute{\mathscr{F}}^U_t)_{t\in\mathbb{R}_0^+}$–martingale under $\acute{P}_U^\nu$ for any $f \in C_K^2(U)$, where $\acute{\mathscr{F}}^U_t := \sigma\{\acute{X}^U_{s\wedge\tau_U} : s \in [0,t]\}$. That is, the stopped process corresponding to a Feller diffusion satisfies a martingale property as in Definition 2.10 (ii). Conversely, we have shown in the previous section that any diffusion in the sense of Definition 2.27 is a Feller process as well as a strong Markov process. Thus, all these basic properties are shared by both classes of diffusion processes. In view of that, it may be justified to refer to processes from both classes as diffusion processes. However, there are considerable differences between Feller diffusions in U and diffusions as in Definition 2.27. To begin with, Feller diffusions in U and diffusions are defined on different spaces. Moreover, Feller diffusions in U do not exist anymore after τ_U, whereas diffusions in the sense of Definition 2.27 are identical Δ from τ_U on. An important question is under which assumptions the respective diffusion processes exist. Recall that a Feller diffusion in U is obtained by killing a Feller diffusion in $\mathbb{R}^d$, and consequently the coefficients a_{ij}^U and b_i^U; $i, j = 1, \dots, d$; appendant to the killed process have limits in $\mathbb{R}$ as $x \to \partial U$, and $(b_i^U)_{i=1,\dots,d}$ is continuous. However, even if the limits $\lim_{x\to\partial U} a_{ij}(x)$ and $\lim_{x\to\partial U} b_i(x)$ do not exist in $\mathbb{R}$ for some $i, j \in \{1, \dots, d\}$, or if $(b_i)_{i=1,\dots,d}$ is not continuous, there may be a diffusion with respect to $T = \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$ in the sense of Definition 2.27.

A possible extension of our considerations in the present thesis may be to deal with diffusions conditional on not leaving U. These conditional processes, which are examples of Doob's h processes, are closely related to the questions with which we will deal in Part II of this thesis. Since the drift coefficients of those conditional diffusions are unbounded on U, these processes are not Feller diffusions in U. However, those conditional processes are diffusions in the sense of Definition 2.27 (cf. the theorem in [Pin85]). Despite not dealing with conditional diffusions in this thesis, our motivation to construct diffusions via the martingale problem is that it enables us to consider those conditional processes in the context of diffusion processes. This may justify the consideration of the approach via the martingale problem in this thesis, even though in Part II we won't take full advantage of the power of the martingale problem.

In Part II of this thesis we will consider diffusion semigroups as given by Definition 2.27.

Part II

Properties of Diffusion Semigroups

Chapter 3

Intrinsic Ultracontractivity

This chapter, in which we will deal with a certain analytical property of diffusion semigroups, constitutes the main part of this thesis. The diffusions are given by Definition 2.27, and some of their properties are provided by the previous chapter.

We consider $\mathbb{R}^d$, $d \geq 3$, endowed with the Euclidean norm $\|\cdot\|_2$. Throughout this chapter let $U \subseteq \mathbb{R}^d$ be a bounded $C^{2,1}$–domain, and let the Borel σ–algebra on U be denoted by $\mathscr{B}(U)$. Additionally, let $\Delta \in \mathbb{R}^d \setminus U$ be a point which yields an Alexandroff one-point compactification $\hat{U} := U \cup \{\Delta\}$ of U (cf. Theorem B.24). Furthermore, we define a function $\sigma_U : C(\mathbb{R}_0^+, \hat{U}) \to \bar{\mathbb{R}}_0^+$ by $\sigma_U(f) = \inf\{t \in \mathbb{R}_0^+ : f(t) \notin U\}$ for all $f \in C(\mathbb{R}_0^+, \hat{U})$. Consider a measurable space $(\Omega, \mathscr{F})$, where $\Omega := \{\omega \in C(\mathbb{R}_0^+, \hat{U}) : \omega(t) = \Delta\, \forall\, t \geq \sigma_U(\omega)\}$ is equipped with the topology of uniform convergence on bounded intervals of $\mathbb{R}_0^+$, and $\mathscr{F} := \mathscr{B}(\Omega)$. Moreover, $\mathscr{B}(\Omega)$ denotes the Borel σ–algebra on Ω. In addition, we will be concerned with the coordinate mapping process $X := (X_t)_{t\in\mathbb{R}_0^+}$ on $(\Omega, \mathscr{F})$, defined by $X_t := \pi_t : \Omega \to \hat{U}$, where π_t denotes the projection as in Definition B.5. Moreover, we consider a filtration $\mathscr{F}^X := (\mathscr{F}_t)_{t\in\mathbb{R}_0^+}$, where $\mathscr{F}_t := \sigma(X_s : s \in [0,t]\}$ for any $t \in \mathbb{R}_0^+$, and we obtain an $\mathscr{F}^X$–stopping time $\tau_U := \sigma_U|_\Omega$. Furthermore, we denote by λ_d the d–dimensional Lebesgue measure and by $\mathrm{ca}(\mathscr{B}(U))$ the family of all finite signed measures on $\mathscr{B}(U)$. In this chapter we will consider diffusions in U, as constructed in the previous chapter (cf. Definition 2.27). Recall that by definition a diffusion in U is a stochastic process killed at τ_U, i.e., with the zero Dirichlet boundary condition.

As the title of this chapter reveals, we will concern ourselves with the so–called intrinsic ultracontractivity. In Section 3.1 we will provide the basic assumptions and tools for our further proceeding in the following sections. The main part of this chapter is Section 3.2, where we will introduce the concept of intrinsic ultracontractivity, and in which we will state and prove the main result of this thesis.

Without going into detail, we just want to mention here the main result of this chapter, so that the reader who is already familiar with the underlying concepts can get a general idea of what we are going to do.

Our main result states that a diffusion semigroup $(P_t)_{t\in\mathbb{R}_0^+}$, as defined in Definition 2.27, is intrinsically ultracontractive on U [with respect to λ_d] if the diffusion coefficient $(a_{ij})_{i,j=1,\ldots,d}$

and the drift coefficient $(b_i)_{i=1,\dots,d}$ satisfy the following conditions:

(i) $a_{ij} \in C^2(\bar{U})$ and $b_i \in C^1(\bar{U})$ for all $i, j \in \{1, \dots, d\}$,

(ii) a_{ij}; $i, j = 1, \dots, d$; can be extended to some $\check{a}_{ij} \in C_b(\mathbb{R}^d)$ such that $\frac{\partial}{\partial x_j}\check{a}_{ij}$ exists and is bounded, and such that $(\check{a}_{ij})_{i,j=1,\dots,d}$ is symmetric as well as uniformly elliptic.

Moreover, if there exists some inner regular finite measure m_0 on $(U, \mathscr{B}(U))$ such that for any $t > 0$ and all $x \in U$ the transition measure $P_t(x, \cdot)$ possesses a bounded continuous m_0–density $p_t^{m_0}(x, \cdot)$, then the first condition can be relaxed to

$$\exists\, \theta > 0\, \forall\, i, j \in \{1, \dots, d\} : a_{ij}, b_i \in C^{0,\theta}(\bar{U})$$

in order that $(P_t)_{t\in\mathbb{R}_0^+}$ is intrinsically ultracontractive on U with respect to m_0. We will present the proof in Section 3.2.

The main literature to which we will refer troughout this chapter are [KS06a], [GQZ88] and [Dyn65II]. Furthermore, in order to prove our main theorem we will utilise results from [HS82] and [GW82]. We will primarily refer to [KS06a] in order to compare the main result obtained by Kim and Song with our main result. However, the methods in that paper are not utilised in the course of our approach. In contrast, [GQZ88] contains some results which are indispensable for our considerations, and we will refer to these results occasionally. Note that the assumptions on the coefficients of the underlying operator in [GQZ88] look slightly different from our assumptions on T, but the proofs of the cited results work out equally well under our assumptions. In fact, the principal part of the diffusion generating operator in [GQZ88] is given in divergence form, whereas our operator will be given in nondivergence form, which may explain the differences.

3.1 Preliminaries

Throughout the whole chapter we consider a linear operator $T : C_K^2(U) \to C_K(U)$ in nondivergence form defined by

$$\forall\, f \in C_K^2(U)\, \forall\, x \in U : Tf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} f(x)\,, \tag{3.1}$$

where $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ satisfy the following conditions:

(i) $\exists\, \theta > 0\, \forall\, i, j \in \{1, \dots, d\} : a_{ij}, b_i \in C^{0,\theta}(\bar{U})$,

(ii) a_{ij}; $i, j = 1, \dots, d$; can be extended to some $\check{a}_{ij} \in C_b(\mathbb{R}^d)$ such that $\frac{\partial}{\partial x_j}\check{a}_{ij}$ exists and is bounded, and such that $(\check{a}_{ij})_{i,j=1,\dots,d}$ is symmetric as well as uniformly elliptic.

Note that $\bar{U}$ denotes the closure of U in $\mathbb{R}^d$. In particular, $(a_{ij})_{i,j=1,\dots,d}$ and $(b_i)_{i=1,\dots,d}$ satisfy the conditions of Theorem 2.19, i.e., for any $\nu \in \mathcal{P}(U)$ the martingale problem for (T, ν) on U has a unique solution $P^\nu \in \mathcal{P}(\Omega, \mathscr{F})$. As in the previous chapter, we set $P^x := P^{\delta_x}$.

Furthermore, let $(P_t)_{t\in\mathbb{R}_0^+}$ be a semigroup of sub–Markov kernels P_t, $t>0$, on $(U,\mathscr{B}(U))$, defined by $P_t(x,B)=P^x(X_t\in B)$ for all $x\in U$ and $B\in\mathscr{B}(U)$. In addition, we define a Feller semigroup $(T_t)_{t\in\mathbb{R}_0^+}$ of transition operators T_t on $C_0(U)$ by $T_tf(x)=\int_U P_t(x,\mathrm{d}y)f(y)$ for all $f\in C_0(U)$ and any $t>0$.

Note that the conditions which we have imposed on the domain U as well as on the diffusion and drift coefficients of T are actually much stronger than the assumptions of Theorem 2.19, which yield a unique solution to the corresponding martingale problem. However, as we will see below, these conditions are necessary in order to establish the set–up for Section 3.2, in which we will utilise the results developed in the present section.

Theorem 3.1 *There exists a positive function $p_{(\cdot)}(\cdot,\cdot)\in C(\mathbb{R}^+\times U\times U)$ such that p_t is bounded and $p_t(x,\cdot)$ is a $\lambda_d|_U$–density of $P_t(x,\cdot)$ for any $t>0$ and all $x\in U$.*

Proof Except for the positivity, this follows from Theorem 0.6 in §6 of the appendix of [Dyn65II].

Our approach to prove the positivity is based on an idea of the proof of Theorem 4.3 in Chapter 2 in [PS78], where Port and Stone prove the positivity of a $\lambda_d|_U$–density relating to the d–dimensional Wiener process with the zero Dirichlet boundary condition.

For all $i,j\in\{1,\dots,d\}$ extend a_{ij} and b_i to some $\check{a}_{ij}\in C_b(\mathbb{R}^d)$ and some bounded measurable $\check{b}_i:\mathbb{R}^d\to\mathbb{R}$ such that $\frac{\partial}{\partial x_j}\check{a}_{ij}$ exists and is bounded, and such that $(\check{a}_{ij})_{i,j=1,\dots,d}$ is symmetric as well as uniformly elliptic. Note that by condition (ii) in the definition of T such extensions are possible, and observe that $(\check{a}_{ij})_{i,j=1,\dots,d}$ and $(\check{b}_i)_{i=1,\dots,d}$ satisfy the assumptions of Theorem 2.4. Moreover, these extensions enable us to use Theorem 1 in [Aro67]. In addition, consider the linear operator $\check{T}:=\frac{1}{2}\sum_{i,j=1}^d\check{a}_{ij}\frac{\partial^2}{\partial x_ix_j}+\sum_{i=1}^d\check{b}_i\frac{\partial}{\partial x_i}$ from $C_K^2(\mathbb{R}^d)$ to $\mathcal{B}(\mathbb{R}^d)$. Then we infer from Theorem 2.4 that for any $x\in\mathbb{R}^d$ the martingale problem for $(\check{T},\delta_x)$ on $\mathbb{R}^d$ has a unique solution $\check{P}^x$ on $(\check{\Omega},\check{\mathscr{F}})$, where $\check{\Omega}:=C(\mathbb{R}_0^+,\mathbb{R}^d)$ and $\check{\mathscr{F}}:=\mathscr{B}(\check{\Omega})$. Here $\check{\Omega}$ is presumed to be endowed with the topology of uniform convergence on bounded intervals of $\mathbb{R}_0^+$, and $\mathscr{B}(\check{\Omega})$ denotes the Borel–σ–algebra on $\check{\Omega}$. Furthermore, we will be concerned with the coordinate mapping process $\check{X}:=(\check{X}_t)_{t\in\mathbb{R}_0^+}$ on $(\check{\Omega},\check{\mathscr{F}})$, defined by $\check{X}_t:=\pi_t:\check{\Omega}\to\mathbb{R}^d$. We will use the random variable $\check{\tau}_U:=\inf\{t\in\mathbb{R}_0^+:\check{X}_t\notin U\}$. Consider the sub–Markov semigroup $(\check{P}_t)_{t\in\mathbb{R}_0^+}$ of transition kernels defined by $\check{P}_t(x,B)=\check{P}^x(\check{X}_t\in B)$ for all $t\in\mathbb{R}_0^+$, $x\in\mathbb{R}^d$ and $B\in\mathscr{B}(\mathbb{R}^d)$, where $\mathscr{B}(\mathbb{R}^d)$ denotes the Borel–σ–algebra on $\mathbb{R}^d$. According to Lemma 2.9 $\check{P}_t(x,\cdot)$, $x\in\mathbb{R}^d$, possesses a λ_d–density $\check{p}_t(x,\cdot)$, which by means of Aronson's estimate (cf. Theorem 1 in [Aro67]) satisfies

$$\alpha^{-1}t^{-d/2}\exp(-\alpha\|x-y\|_2^2t^{-1})\le\check{p}_t(x,y)\le\alpha t^{-d/2}\exp(-\alpha^{-1}\|x-y\|_2^2t^{-1})\qquad(3.2)$$

for all $y\in\mathbb{R}^d$.

<u>Part I</u>

Let $x_0\in U$. Based on the proof of Proposition 4.1 in Chapter 2 in [PS78] we will show that there exists some $\varepsilon>0$ such that $p_t(x,y)\check{p}_t(x,y)^{-1}\to1$ uniformly in $x,y\in B[x_0,\varepsilon]$ as $t\to0$. For any $t>0$ consider $p'_t:U\times U\to\mathbb{R}_0^+$ given by $p'_t(x,y)=\check{p}_t(x,y)-p_t(x,y)$ for all

$x, y \in U$, and observe that

$$\forall\, x, y \in U : p'_t(x,y) = \int_{\{\check{\tau}_U < t\}} \check{p}_{t-\check{\tau}_U}(X_{\check{\tau}_U}(\omega), y)\check{P}^x(\mathrm{d}\omega). \tag{3.3}$$

Indeed, since $\check{P}^x(\check{X}_t \in B, \check{\tau}_U = t) \le \check{P}^x(\check{X}_t \in B, \check{X}_t \in \partial U) = 0$ for all $B \in \mathscr{B}(U)$, we have that

$$\begin{aligned}\int_B \check{p}_t(x,y)\lambda_d(\mathrm{d}y) &= \check{P}^x(\check{X}_t \in B) = \check{P}^x(\check{X}_t \in B, \check{\tau}_U > t) + \check{P}^x(\check{X}_t \in B, \check{\tau}_U < t)\\ &= P^x(X_t \in B, \check{\tau}_U > t) + \int_{\{\check{\tau}_U < t\}} \check{P}^{\check{X}_{\check{\tau}_U}(\omega)}(\check{X}_{t-\check{\tau}_U} \in B)(\cdot, y)\check{P}^x(\mathrm{d}\omega)\\ &= \int_B p_t(x,y)\lambda_d(\mathrm{d}y) + \int_B \int_{\{\check{\tau}_U < t\}} \check{p}_{t-\check{\tau}_U}(X_{\check{\tau}_U}(\omega), y)\check{P}^x(\mathrm{d}\omega)\lambda_d(\mathrm{d}y)\end{aligned}$$

holds true for all $x, y \in U$ and each $B \in \mathscr{B}(U)$. By the continuity on U of the integrands of the integrals above, this results in

$$\forall\, x, y \in U : \check{p}_t(x,y) = p_t(x,y) + \int_{\{\check{\tau}_U < t\}} \check{p}_{t-\check{\tau}_U}(X_{\check{\tau}_U}(\omega), y)\check{P}^x(\mathrm{d}\omega),$$

which shows (3.3). Let $\rho := \inf\{\|z - x_0\| : z \in U^{\complement}\}$ and fix some $\varepsilon > 0$ with $3\varepsilon < \rho$. Furthermore, let $x, y \in B[x_0, \varepsilon]$ and define $\eta := \inf\{\|z - v\| : z \in U^{\complement}, v \in B[x_0, \varepsilon]\}$. Note that $\|x - y\|_2 \le 2\varepsilon < \rho - \varepsilon \le \eta$. In addition, observe that $\alpha t^{-d/2} \exp(-\alpha^{-1}\eta^2 t^{-1})$ is monotonically increasing in $t \in (0, 2\eta^2\alpha^{-1}d^{-1})$. Therefore, we deduce from (3.2) that

$$\begin{aligned}\check{p}_{t-s}(z,y) &\le \alpha(t-s)^{-d/2}\exp(-\alpha^{-1}\|z-y\|_2^2(t-s)^{-1})\\ &\le \alpha(t-s)^{-d/2}\exp(-\alpha^{-1}\eta^2(t-s)^{-1}) \le \alpha t^{-d/2}\exp(-\alpha^{-1}\eta^2 t^{-1}).\end{aligned}$$

for all $0 \le s < t < 2\eta^2\alpha^{-1}d^{-1}$ and every $z \in U^{\complement}$. By means of (3.3) this yields that

$$\forall\, 0 < t < 2\eta^2\alpha^{-1}d^{-1} : p'_t(x,y) \le \alpha t^{-d/2}\exp(-\alpha^{-1}\eta^2 t^{-1}),$$

which, by (3.2), results in

$$\forall\, 0 < t < 2\eta^2\alpha^{-1}d^{-1} : \frac{p'_t(x,y)}{\check{p}_t(x,y)} \le \exp\left(-\frac{\eta^2 - \|x-y\|_2^2}{\alpha t}\right) \le \exp\left(-\frac{\eta^2 - 4\varepsilon^2}{\alpha t}\right).$$

Since $p'(_t x, y) = \check{p}_t(x,y) - p_t(x,y)$, we conclude by the inequality above that

$$p_t(x,y)\check{p}_t(x,y)^{-1} \to 1 \text{ uniformly in } x, y \in B[x_0, \varepsilon] \text{ as } t \to 0.$$

<u>Part II</u>

Now fix some arbitrary $t_0 > 0$ and $x \in U$, and define $A := \{y \in U : p_{t_0}(x,y) > 0\}$, which is open in U by the continuity of $p_{t_0}(x, \cdot)$. Observe that we obtain $A = U$ if we show that $A \neq \emptyset$ and $\bar{A} \cap U = A$. This can be seen as follows: Since $A \subseteq U$ by definition, we only have to show that $U \subseteq A$. From $\bar{A} \cap U = A$ we infer that $U \cap \partial A = \emptyset$. Because U is connected and $A \neq \emptyset$, this yields that $U \cap A^{\complement} = \emptyset$, which results in $A = U$.

At first observe that $A \neq \emptyset$. In fact, $P_{t_0}(x, U) = P_{t_0}(x, A)$, since $P_{t_0}(x, U \setminus A) = 0$ by definition of A. Moreover, since $x \in U$ and because the diffusion is continuous, we have that $P_{t_0}(x, U) > 0$. Now if $A = \emptyset$, then we had $P_{t_0}(x, U) = 0$, which would be a contradiction to the positivity of $P_{t_0}(x, U)$.

Since $A \subseteq \bar{A} \cap U$, it only remains to show that $\bar{A} \cap U \subseteq A$.

For the time being, let $y \in U$ and observe that there exists a neighbourhood $N_y \subseteq U$ of y such that $p_t(u, v) > 0$ for all $u, v \in N_y$ and every $t > 0$. This can be seen as follows: By Part I there exists an $\varepsilon > 0$ such that $p_t(u, v)\check{p}_t(u, v)^{-1} \to 1$ uniformly in $u, v \in B[y, \varepsilon]$ as $t \to 0$. With $N_y := B[y, \varepsilon]$ we infer that there exists some $t_y > 0$ such that $p_t(u, v) > 0$ for all $u, v \in N_y$ and any $t \leq t_y$. Now let $t > 0$ and choose some $n_t \in \mathbb{N}$ with $n_t t_y \leq t \leq (n_t + 1)t_y$, i.e., $0 \leq t - n_t t_y \leq t_y$. Now the Chapman–Kolmogorov equation yields that

$$\begin{aligned} p_t(u, v) &= \int_U p_{n_t t_y}(u, w) p_{t - n_t t_y}(w, v) \lambda_d(\mathrm{d}w) \\ &= \int_U \underbrace{\int_U \ldots \int_U}_{n_t \text{ times}} p_{t_y}(u, w_1) \ldots p_{t_y}(w_{n_t}, w) p_{t - n_t t_y}(w, v) \lambda_d(\mathrm{d}w_1) \ldots \lambda_d(w_{n_t}) \lambda_d(\mathrm{d}w) > 0 \end{aligned}$$

holds true for all $u, v \in N_y$.

Choose some $z \in \bar{A} \cap U$. Then, as shown above, there exists a neighbourhood $N_z \subseteq U$ of z such that $p_t(u, v) > 0$ for all $u, v \in N_z$ and every $t > 0$. Note that $A \cap N_z \neq \emptyset$, since $z \in \bar{A}$. Let $y \in A \cap N_z$ and observe that $p_{t_0}(x, y) > 0$, because $y \in A$. By the continuity of $p_{(\cdot)}(x, y)$ there exists an $0 < s < t_0$ with $p_s(x, y) > 0$. Since $p_s(x, \cdot)$ is continuous, we conclude that there exists a neighbourhood $N_y \subseteq N_z$ of y with $p_t(x, w) > 0$ for all $w \in N_y$. Another application of the Chapman–Kolmogorov equation in conjunction with our above considerations results in

$$p_{t_0}(x, z) = \int_U p_s(x, w) p_{t_0 - s}(w, z) \lambda_d(\mathrm{d}w) \geq \int_{N_y} p_s(x, w) p_{t-s}(w, z) \lambda_d(\mathrm{d}v) > 0,$$

i.e., $z \in A$. Since $z \in \bar{A} \cap U$ was chosen arbitrarily, we infer that $\bar{A} \cap U \subseteq A$. As mentioned above, this yields the assertion. □

Throughout this chapter let $p_{(\cdot)}(\cdot, \cdot)$ denote the function as in Theorem 3.1.

Remark 3.2 (i) Note that in the proof of Theorem 3.1 we can indeed apply Theorem 0.6 in §6 of the appendix of [Dyn65II], because we can extend $(a_{ij})_{i,j=1,\ldots,d}$ and $(b_i)_{i=1,\ldots,d}$ to coefficients on some domain $D \supseteq \bar{U}$ which satisfy the assumptions of Theorem 2.19. Since all trajectories ω_D of the corresponding diffusion (cf. Definition 2.27) are continuous up to $\inf\{t \in \mathbb{R}_0^+ : \omega_D(t) \notin D\}$, we deduce by means of Theorem 2.2 that for every $\omega \in \Omega$ there exists some $x_0 \in \partial U$ such that $\|\omega(t) - x_0\|_2 \to 0$ as $t \to \tau_U(\omega)$.

(ii) Observe that $(p_t)_{t\in\mathbb{R}^+}$ satisfies the Chapman–Kolmogorov equation with respect to λ_d. Indeed, we know that $(P_t)_{t\in\mathbb{R}_0^+}$ is a semigroup. Thus we infer that

$$\begin{aligned}\int_B p_{s+t}(x,y)\lambda_d(\mathrm{d}y) &= P_{s+t}(x,B) = \int_U P_s(x,\mathrm{d}z)P_t(z,B) \\ &= \int_B \int_U p_s(x,z)p_t(z,y)\lambda_d(\mathrm{d}z)\lambda_d(\mathrm{d}y)\end{aligned}$$

holds true for all $x \in U$, $B \in \mathscr{B}(U)$ and $s,t>0$, i.e., $p_t(x,y) = \int_U p_s(x,z)p_t(z,y)\lambda_d(\mathrm{d}z)$. $\Diamond$

Later on we will frequently deal with an "adjoint" semigroup of the Feller semigroup $(T_t)_{t\in\mathbb{R}_0^+}$. Actually, rather than considering the adjoint semigroup itself, we will consider a semigroup which is closely related to the adjoint semigroup, and which will be referred to as formal adjoint semigroup.

We define a semigroup $(U_t)_{t\in\mathbb{R}_0^+}$ of linear operators $U_t : \mathrm{ca}(\mathscr{B}(U)) \to \mathrm{ca}(\mathscr{B}(U))$ by

$$\forall\, \nu \in \mathrm{ca}(\mathscr{B}(U))\,\forall\, B \in \mathscr{B}(U) : U_t\nu(B) = \int_U P_t(x,B)\nu(\mathrm{d}x).$$

The dual space $\mathcal{B}(U)^*$ of $\mathcal{B}(U)$ is related to $\mathrm{ca}(\mathscr{B}(U))$ in the following way: Each $f^* \in \mathcal{B}(U)^*$ can be represented by

$$\forall\, f \in \mathcal{B}(U) : f^*(f) = \int_U f \;\mathrm{d}\nu =: \langle f, \nu\rangle,$$

where $\nu \in \mathrm{ca}(\mathscr{B}(U))$. That is, any $f^* \in \mathcal{B}(U)^*$ corresponds to some $\nu \in \mathrm{ca}(\mathscr{B}(U))$, and thus $\mathcal{B}(U)^*$ can be equated with $\mathrm{ca}(\mathscr{B}(U))$. Note that

$$\langle T_t f, \nu\rangle = \int_U\int_U p_t(x,y)f(y)\lambda_d(\mathrm{d}y)\nu(\mathrm{d}x) = \int_U f(y)\int_U P_t(x,\mathrm{d}y)\nu(\mathrm{d}x) = \langle f, U_t\nu\rangle$$

holds true for all $t>0$, $f \in C_0(U)$ and $\nu \in \mathrm{ca}(\mathscr{B}(U))$, i.e., U_t is the adjoint of T_t. Moreover, observe that

$$\forall\, t>0\,\forall\, \nu \in \mathrm{ca}(\mathscr{B}(U))\,\forall\, B \in \mathscr{B}(U) : U_t\nu(B) = \int_B\int_U p_t(x,y)\nu(\mathrm{d}x)\lambda_d(\mathrm{d}y),$$

i.e., for any $\nu \in \mathrm{ca}(\mathscr{B}(U))$ the signed measure $U_t\nu$ has a $\lambda_d|_U$–density $\int_U p_t(x,\cdot)\nu(\mathrm{d}x) \in C_b(U)$.

We define a semigroup $(T_t^*)_{t\in\mathbb{R}_0^+}$ of linear operators $T_t^* : C_b(U) \to C_b(U)$ by

$$\forall\, t>0\,\forall\, f \in C_b(U) : T_t^* f = \int_U p_t(x,\cdot)f(x)\lambda_d(\mathrm{d}x).$$

In particular, we obtain for any $f \in C_b(U)$ that $T_t^* f$ is a $\lambda_d|_U$–density of $U_t(f\lambda_d)$, since

$$\int_B T_t^* f(y)\lambda_d(\mathrm{d}y) = \int_B\int_U p_t(x,y)f(x)\lambda_d(\mathrm{d}x)\lambda_d(\mathrm{d}y) = U_t(f\lambda_d)(B) \tag{3.4}$$

holds for all $t > 0$, $B \in \mathscr{B}(U)$ and $y \in U$. Furthermore, we infer that

$$\int_U f(x)T_t^* g(x)\lambda_d(\mathrm{d}x) = \int_U \int_U f(x)p_t(y,x)g(y)\lambda_d(\mathrm{d}y)\lambda_d(\mathrm{d}x) = \int_U g(y)T_t f(y)\lambda_d(\mathrm{d}y) \quad (3.5)$$

holds true for all $f \in C_0(U)$, $g \in C_b(U)$ and $t > 0$. We call $(T_t^*)_{t\in\mathbb{R}_0^+}$ the **formal adjoint semigroup** of $(T_t)_{t\in\mathbb{R}_0^+}$. A motivation for this name will be given in Lemma 3.14 on page 60. Let $(P_t^*)_{t\in\mathbb{R}_0^+}$ denote the semigroup of transition kernels associated with $(T_t^*)_{t\in\mathbb{R}_0^+}$.

Throughout this chapter we will make frequent use of the following result concerning eigenvalues and eigenfunctions, which is based on the strong version of the Krein–Rutman theorem (cf. [KR62]) and the strong maximum principle (cf. [GT98]).

Theorem 3.3 *There exists a $\gamma > 0$ such that $e^{-\gamma t}$ is a simple eigenvalue of T_t and T_t^* for any $t > 0$. Furthermore, $|l| < e^{-\gamma t}$ for all $t > 0$ and $l \in (\sigma(T_t) \cup \sigma(T_t^*)) \setminus \{e^{-\gamma t}\}$. Moreover, there exist positive $\varphi, \psi \in C_0(U)$, normalised such that $\int_U \psi \, d\lambda_d = 1$ and $\int_U \varphi\psi \, d\lambda_d = 1$, which satisfy $T_t\varphi = e^{-\gamma t}\varphi$ as well as $T_t^*\psi = e^{-\gamma t}\psi$ for all $t > 0$.*

Proof See Theorem 5.5 and Theorem 6.1 in Chapter 3 in [Pin95] as well as Proposition 3 and Proposition 4 in [GQZ88]. Fix some $t > 0$, and note that we don't get directly the result concerning T_t^* and ψ. But since U_t is the adjoint of T_t, we obtain that there exists a measure ν on $\mathscr{B}(U)$, normalised such that $\nu(U) = 1$ and $\int_U \varphi \, \mathrm{d}\nu = 1$, with $U_t\nu = e^{-\gamma t}\nu$ and such that $\nu(B) > 0$ for any nonempty open $B \subseteq U$. Let $\psi \in C_b(U)$ be a $\lambda_d|_U$–density of ν. Note that such a $\psi \in C_b(U)$ exists, since, as we have shown above, $U_t\nu$ has a $\lambda_d|_U$–density in $C_b(U)$. According to (3.4) we have that

$$\int_B T_t^*\psi \, \mathrm{d}\lambda_d = U_t(\psi\lambda_d)(B) = U_t\nu(B) = e^{-\gamma t}\nu(B) = e^{-\gamma t}\int_B \psi \, \mathrm{d}\lambda_d$$

holds true for all $B \in \mathscr{B}(U)$, and hence $T_t^*\psi = e^{-\gamma t}\psi$. That ψ is positive follows from the positivity of p_t and from $\nu(B) > 0$ for all nonempty open $B \subseteq U$. Moreover, that $\int_U \psi \, \mathrm{d}\lambda_d = 1$ and $\int_U \varphi\psi \, \mathrm{d}\lambda_d = 1$ results from $\nu(U) = 1$ and $\int_U \varphi \, \mathrm{d}\nu = 1$, because ψ is a $\lambda_d|_U$–density of ν. □

Remark 3.4 Note that the Krein–Rutman theorem is not applicable directly, because $C_0(U)$ has an empty interior, and the strong version of the Krein–Rutman theorem requires that the cone of the nonnegative functions in the respective function space has a nonempty interior. Thus, this theorem cannot be applied directly to an operator on $C_0(U)$. However, in Section 5 of Chapter 3 in [Pin95] Pinsky shows a way of how to apply the Krein–Rutman theorem in order to prove Theorem 3.3. ◊

Throughout this chapter let γ, φ and ψ denote the constant and eigenfunctions, respectively, as in Theorem 3.3.

In the course of our considerations we will frequently refer to the following hypotheses:

(H1) There exists some inner regular $m_0 \in \mathcal{M}_f(U)$ such that for any $t > 0$ and all $x \in U$ the transition measure $P_t(x, \cdot)$ possesses a bounded, continuous and symmetric m_0–density $p_t^{m_0}(x, \cdot)$, i.e., $P_t(x, B) = \int_B p_t(x, y)m_0(\mathrm{d}y)$ and $p_t^{m_0}(x, \cdot) = p_t^{m_0}(\cdot, x)$ for all $x \in U$, any $t > 0$ and every $B \in \mathscr{B}(U)$.

(H1') We say that a finite inner regular measure m on $(U, \mathscr{B}(U))$ satisfies Hypothesis **(H1')** if for any $t > 0$ and all $x \in U$ the transition measure $P_t(x, \cdot)$ possesses a positive, bounded, continuous and symmetric m–density $p_t^m(x, \cdot)$.

(H2) In addition to the assumptions on the coefficients in the definition of T we have that $a_{ij} \in C^2(\bar{U})$ and $b_i \in C^1(\bar{U})$ for all $i, j \in \{1, \ldots, d\}$.

We do not postulate that Hypothesis **(H1)** or Hypothesis **(H2)** is satisfied, but for some results in the upcoming part of this chapter we will assume that **(H1)** or **(H2)** is satisfied. In these situations we will explicitely refer to the corresponding hypothesis.

Observe that if Hypothesis **(H1)** is satisfied, then $p_t^{m_0}$ satisfies the Chapman–Kolmogorov equation with respect to m_0 (cf. Remark 3.2 (ii)).

Note that Hypothesis **(H1)** is satisfied whenever there exists a measure $m \in \mathcal{M}_f(U)$ which satisfies Hypothesis **(H1')**.

In particular, Hypothesis **(H1)** is satisfied if the diffusion is symmetric. In fact, in that situation we deduce from Theorem 3.1 that λ_d satisfies Hypothesis **(H1')**. However, Hypothesis **(H1)** does not require the diffusion to be symmetric.

In particular, Hypothesis **(H2)** implicates that $\frac{\partial^2}{\partial x_i \partial x_j} a_{ij}$ as well as $\frac{\partial}{\partial x_i} b_i$ are bounded for all $i, j \in \{1, \ldots, d\}$.

3.2 Intrinsic Ultracontractivity

Now we are ready to introduce the main concept of this thesis, the so–called intrinsic ultracontractivity. As we will see in Chapter 4, this property is sufficiently strong in order to guarantee nice convergence properties of the diffusion. In fact, that is our main motivation to deal with intrinsic ultracontractivity.

Definition 3.5 *Let m be a finite measure on $(U, \mathscr{B}(U))$, such that for any $t > 0$ as well as each $x \in U$ the transition measure $P_t(x, \cdot)$ has an m–density $p_t^m(x, \cdot)$. We say that the diffusion semigroup $(P_t)_{t \in \mathbb{R}_0^+}$ is* **intrinsically ultracontractive** *[on U] with respect to m if for any $t > 0$ the function $q_t^m : U \times U \to \mathbb{R}_0^+$, defined by*

$$\forall\, t > 0\, \forall\, x, y \in U : q_t^m(x, y) = \frac{p_t^m(x, y)}{\varphi(x)\psi(y)}, \tag{3.6}$$

is bounded.

If $m = \lambda_d$ (i.e., $p^m = p$) and if (3.6) is satisfied, then we will simply say $(P_t)_{t \in \mathbb{R}_0^+}$ is intrinsically ultracontractive [on U], without mentioning the measure.

The concept of intrinsic ultracontractivity for symmetric processes was introduced by Davies and Simon in [DS84]. Davies and Simon used a different definition (cf. Section 3 in [DS84]). They considered a semigroup $(L_t)_{t \in \mathbb{R}_0^+}$ of transition operators on $\mathscr{L}^2(U, \lambda_d|_U)$ with associated semigroup $(Q_t)_{t \in \mathbb{R}_0^+}$ of transition kernels and normalised eigenfunction φ corresponding to the principal eigenvalue $e^{-\gamma t}$, i.e., $L_t\varphi = e^{-\gamma t}\varphi$ for all $t > 0$. They call $(L_t)_{t\in\mathbb{R}_0^+}$ intrinsically ultracontractive if for any $t > 0$ the operator L'_t on $\mathscr{L}^2(U, \varphi^2\lambda_d|_U)$, defined by $L'_t f(x) = \int_U Q_t(x, \mathrm{d}y)\varphi(x)^{-1}\varphi(y)^{-1}f(y)$ for all $f \in \mathscr{L}^2(U, \varphi^2\lambda_d|_U)$ and $x \in U$, is a bounded operator which maps $\mathscr{L}^2(U, \varphi^2\lambda_d|_U)$ into $\mathscr{L}^\infty(U, \varphi^2\lambda_d|_U)$. In Theorem 3.2 in [DS84] Davies and Simon show that in this symmetric $\mathscr{L}^2$–context their definition is equivalent to our definition. Moreover, in Proposition 2.2 in [KS06a] Kim and Song prove that the corresponding equivalence holds true also for non–symmetric operators.

All our considerations in this chapter are related to the bounded $C^{2,1}$–domain U. However, in the literature, many authors consider a diffusion with "nice" coefficients, e.g. the Wiener process, on some bounded domain $D \subseteq \mathbb{R}^d$ with the zero Dirichlet boundary condition, and they are interested in how "regular" the boundary of D has to be for intrinsic ultracontractivity to hold. That is, whereas our approach is to fix the domain and to find coefficients as general as possible, a common approach in the literature is to fix a "nice" diffusion and to find a domain as general as possible to have intrinsic ultracontractivity. This approach was chosen for example by Davies and Simon in [DS84], where they prove that the semigroup corresponding to the Laplace operator with the zero Dirichlet boundary condition is intrinsically ultracontractive if D has a C^∞–boundary (cf. Theorem 9.2 in [DS84]) or if D satisfies an interior cone condition and an exterior cone condition (cf. Theorem 9.3 in [DS84]). Much more general results in this regard can be derived from Theorem 1.2 in [BB92], where Bass and Burdzy show that on certain domains the parabolic boundary Harnack principle (cf. (3.11) on p. 57) holds for a large class of symmetric operators. We show that the parabolic boundary Harnack principle is equivalent to intrinsic ultracontractivity (cf. Theorem 3.11 on p. 57), and thus they have effectively proven that intrinsic ultracontractivity holds under their assumptions.

Another approach, which was employed by Cipriani in Theorem 3 in [Cip94] as well as by Ouhabaz and Wang in Corollary 2.4 (a) in [OW07], is to investigate under which assumptions on the eigenfunction φ intrinsic ultracontractivity holds true.

Almost all authors considered intrinsic ultracontractivity in the context of symmetric diffusions. The paper closest to our considerations is [KS06a], firstly because Kim and Song are the first (and to the best of our knowledge only ones) who considered intrinsic ultracontractivity of non–symmetric diffusion semigroups, and secondly because they have taken the approach to fix a domain and to consider how the coefficients have to be chosen in order that intrinsic ultracontractivity holds. We are going to deal with a similar question. In view of these similarities of their paper with this thesis, we will compare the main result in [KS06a] with our main result, in order to point out where is the novelty of our work. This comparison is done at the end of the present section. Note that intrinsic ultracontractivity is an analytical property, and so far this property was mainly studied in the Hilbert space context, often using Dirichlet forms and logarithmic Sobolev inequalities. That functional analytical approach was also used by Kim and Song, whereas we are working on $C_0(U)$ and we have chosen a more stochastic approach. That is, our method differs profoundly from the approach in [KS06a].

As we have noted above, many authors are interested in the question under which assumptions intrinsic ultracontractivity holds true. Thus one may wonder if there are situations at all in which intrinsic ultracontractivity does not hold. Well, in Theorem 9.1 in [DS84] Davies and Simon give a counterexample of a bounded domain in $\mathbb{R}^2$ on which the semigroup corresponding to the Laplace operator with the zero Dirichlet boundary condition is not intrinsically ultracontractive. Another counterexample is given in Section 4 in [BD89] by Bañuelos and Davis. There they show that $e^{-\gamma t}p_t(x,y)\varphi(x)^{-1}\psi(y)^{-1}$ does not converge jointly uniformly in $(x,y) \in D^2$ as $t \to \infty$, but intrinsic ultracontractivity implies such a jointly uniformly convergence. These counterexamples, which fit into the approach of considering "nice" diffusions and "irregular" domains, show that, as one may have expected, some regularity conditions on the boundary of D are necessary in order to obtain intrinsic ultracontractivity of the diffusion semigroup. Indeed, it turns out that what might cause problems are what in [BB92] Bass and Burdzy call "long and thin canals".

Above we have mentioned that there exist many results which show that on bounded domains with sufficiently regular boundary intrinsic ultracontractivity holds for "nice" diffusions, in particular for the Wiener process with the zero Dirichlet boundary condition. However, except for the results in [KS06a] these results are related to symmetric operators. As mentioned above, in [KS06a] there is developed an idea for dealing with intrinsic ultracontractivity in the context of non–symmetric diffusions, which differs profoundly from our method. Before we will present our results, we will present a class of diffusions in U whose diffusion semigroups are intrinsically ultracontractive.

Consider the function $\delta : U \to \mathbb{R}^+$ defined by $\delta(x) = \inf_{y\in\partial U} \|x-y\|_2$ for all $x \in U$. This definition will be valid throughout the whole section.

Lemma 3.6 *There exist some $\alpha > \beta > 0$ such that*

$$\beta\delta(x) \le \varphi(x) \le \alpha\delta(x) \quad \textit{and} \quad \beta\delta(x) \le \psi(x) \le \alpha\delta(x)$$

hold true for every $x \in U$.

Proof Confer Proposition 3 in [GQZ88]. □

Example 3.7 *Let $g = (g_i)_{i=1,\dots,d} \in C_b^\infty(U,\mathbb{R}^d)$ be such that $\frac{\partial}{\partial x_i} g$ is bounded. Then $(P_t^g)_{t\in\mathbb{R}_0^+}$, the semigroup of the Wiener process with drift g killed at τ_U, is intrinsically ultracontractive.*

Proof For every $x \in U$ let $p_t^g(x,\cdot)$ denote the $\lambda_d|_U$–density of $P_t^g(x,\cdot)$. Then we have by Theorem 4.2 in [KS06b] that there exist $\alpha_1, \alpha_2 > 0$ such that

$$p_t^g(x,y) \le \alpha_1 t^{-\frac{d+2}{2}}\delta(x)\delta(y)\exp\left(-\alpha_2\frac{\|x-y\|_2^2}{2t}\right)$$

holds for all $x, y \in U$. By means of Lemma 3.6 this results in

$$p_t^g(x,y) \le \alpha\varphi(x)\psi(y)$$

for some $\alpha >$ and all $x, y \in U$, i.e., $(P_t^g)_{t\in\mathbb{R}_0^+}$ is intrinsically ultracontractive. □

Observe that in the above example the generator T^g of the Wiener process with drift g killed at τ_U is given on $C_K^2(U)$ by $T^g f = {}^1\!/\!{}_2\Delta_d f + \sum_{i=1}^d g_i \frac{\partial}{\partial x_i} f(x)$, where Δ_d denotes the d–dimensional Dirichlet Laplacian.

Now we start to develop our theory concerning intrinsic ultracontractivity, which comprises the non–symmetric case. The first result states that the upper bound, which is obtained by intrinsic ultracontractivity, yields a positive lower bound.

Theorem 3.8 *Assume that for every $t > 0$ there exist an $\alpha_t > 0$ and some inner regular $m \in \mathcal{M}(U)$ such that for any $x \in U$ the measure $P_t(x, \cdot)$ possesses an m–density p_t^m, which satisfies $p_t^m(x, y) \leq \alpha_t \varphi(x)\psi(y)$ for all $y \in U$. Then for any $t > 0$ there exists a $\beta_t > 0$ and some $\tilde{m} \in \mathcal{M}_f(U)$, which satisfies Hypothesis* **(H1')**, *such that for each $x \in U$ we have that $p_t^{\tilde{m}}(x, \cdot)$, the corresponding $\tilde{m}$–density of $P_t(x, \cdot)$, satisfies $\beta_t \varphi(x)\psi(y) \leq p_t^{\tilde{m}}(x, y)$ for all $y \in U$.*

Proof Our approach is based on the ideas of the proof of Theorem 3.2 "(iv)⟹(v)" in [DS84]. Let $t > 0$. The inner regularity of m implies that there exists a compact $K_0 \subseteq U$ with $\lambda_d(U \setminus K_0) \leq \alpha_t^{-1} 2^{-1} e^{-\gamma t} [\max_{x \in U} \varphi(x) \max_{x \in U} \psi(x)]^{-1}$, which yields that

$$\int_{U \setminus K_0} \varphi(x)\psi(x) m(\mathrm{d}x) \leq \max_{x \in U} \varphi(x) \max_{x \in U} \psi(x) m(U \setminus K_0) \leq \frac{e^{-\gamma t}}{2\alpha_t}. \tag{3.7}$$

We deduce from $e^{-\gamma t}\varphi(x) = T_t \varphi = \int_U P_t(x, \mathrm{d}y)\varphi(y)$ and $p_t^m(x, z) \leq \alpha_t \varphi(x)\psi(z)$; $x, z \in U$; that

$$\begin{aligned} e^{-\gamma t}\varphi(x) &\leq \alpha_t \varphi(x) \int_{U \setminus K_0} \varphi_1(y)\psi_1(y) m(\mathrm{d}y) + \int_{K_0} P_t(x, \mathrm{d}y)\varphi(y) \\ &\overset{(3.7)}{\leq} \frac{1}{2} e^{-\gamma t}\varphi(x) + \int_{K_0} P_t(x, \mathrm{d}y)\varphi(y). \end{aligned}$$

holds for all $x \in U$. This and an analogous consideration with $(P_s^*)_{s \in \mathbb{R}_0^+}$ result in

$$\frac{1}{2} e^{-\gamma t}\varphi(x) \leq \int_{K_0} P_t(x, \mathrm{d}y)\varphi(y) \quad \text{and} \quad \frac{1}{2} e^{-\gamma t}\psi(x) \leq \int_{K_0} P_t^*(x, \mathrm{d}y)\psi(y) \tag{3.8}$$

for all $x \in U$. Moreover, observe that the continuity and positivity on $U \times U$ and U, respectively, of $p_t^{\tilde{m}}$, φ and ψ implies that $F_t : U \times U \to \mathbb{R}$, defined by $F_t(x, y) = p_t^{\tilde{m}}(x, y)\varphi(x)^{-1}\psi(y)^{-1}$ for all $x, y \in U$, is continuous and positive on $U \times U$, and thus $\kappa_t := \min_{x, y \in K_0} F_t(x, y) > 0$. Now we infer from the Chapman–Kolmogorov equation that

$$\begin{aligned} p_s^{\tilde{m}}(x, y) &= \int_U \int_U p_{s/3}^{\tilde{m}}(x, u) p_{s/3}^{\tilde{m}}(u, v) p_{s/3}^{\tilde{m}}(v, y) \tilde{m}(\mathrm{d}u)\tilde{m}(\mathrm{d}v) \\ &\geq \int_{K_0} \int_{K_0} p_{s/3}^{\tilde{m}}(x, u) p_{s/3}^{\tilde{m}}(u, v) p_{s/3}^{\tilde{m}}(v, y) \tilde{m}(\mathrm{d}u)\tilde{m}(\mathrm{d}v) \\ &\geq \kappa_{s/3} \int_{K_0} P_{s/3}(x, \mathrm{d}u)\varphi(u) \int_{K_0} P_{s/3}^*(y, \mathrm{d}v)\psi(v) \\ &\overset{(3.8)}{\geq} \frac{1}{4} \kappa_{s/3} e^{-\frac{2}{3}\gamma s} \varphi(x)\psi(y) \end{aligned}$$

holds for all $x, y \in U$ and every $s > 0$, which yields the assertion with $\beta_s := \frac{1}{4}\kappa_{s/3} e^{-\frac{2}{3}\gamma s}$. □

The previous theorem will be of interest in particular in Chapter 4, where it is an indispensable tool for our considerations.

The following lemma will be very useful in order to prove intrinsic ultracontractivity.

Lemma 3.9 *Let $t > 0$, and assume there exists a compact $K_t \subseteq U$ and an $\alpha_{t,K_t} > 0$ with $P_{t/2}(\cdot, U) \le \alpha_{t,K_t} P_{t/2}(\cdot, K_t)$. Then there exists some $\kappa_{t,K_t} > 0$ such that $p_t^m(x, \cdot) \le \kappa_{t,K_t}\varphi(x)$ holds for all $x \in U$ and any $m \in \mathcal{M}$ which satisfies that $P_t(x, \cdot)$ has a bounded m–density $p_t^m(x, \cdot)$ for all $x \in U$.*

Proof At first note that $e^{-\gamma t}\varphi = T_t\varphi$ for all $t > 0$ implies that

$$\varphi(x) = e^{\gamma t}\int_U p_t^m(x,y)\varphi(y)\lambda_d(\mathrm{d}y) \ge \int_K p_t^m(x,y)\varphi(y)\lambda_d(\mathrm{d}y) \ge \min_{y\in K}\varphi(y)\int_K p_t^m(x,y)\lambda_d(\mathrm{d}y)$$

holds for all $x \in U$ and any compact $K \subseteq U$, since $e^{\gamma t} > 1$. That is, for every $x \in U$ and each compact $K \subseteq U$ we have

$$\int_K p_t^m(x,y)\lambda_d(\mathrm{d}y) \le \xi_K\varphi(x), \tag{3.9}$$

where $\xi_K := [\min_{y\in K}\varphi(y)]^{-1}$.

Put $\rho_t := \max_{x,y} p_{t/2}^m(x, y)$. Then we deduce by means of the Chapman–Kolmogorov equation and (3.9) that

$$\begin{aligned} p_t^m(x,y) &= \int_U p_{t/2}^m(x,z)p_{t/2}^m(z,y)\lambda_d(\mathrm{d}z) \\ &\le \rho_t\int_U p_{t/2}^m(x,z)\lambda_d(\mathrm{d}z) \\ &\le \rho_t\alpha_{t,K_t}\int_{K_t} p_{t/2}^m(x,z)\lambda_d(\mathrm{d}z) \\ &\le \rho_t\alpha_{t,K_t}\xi_{K_t}\varphi(x) \end{aligned} \tag{3.10}$$

holds true for all $x, y \in U$. With $\kappa_{t,K_t} := \rho_t\alpha_{t,K_t}\xi_{K_t}$ we thus have that $p_t^m(x, \cdot) \le \kappa_{t,K_t}\varphi(x)$ for all $x \in U$. □

Now we are in a position to prove a first equivalent characterisation of intrinsic ultracontractivity in case that Hypothesis **(H1)** is satisfied. The first direction is basically a corollary of Theorem 3.8, whereas the converse direction follows from the previous lemma in conjunction with Lemma 3.6.

Theorem 3.10 *Presume that Hypothesis* **(H1)** *is satisfied, then the following properties are equivalent:*

(i) $(P_t)_{t\in\mathbb{R}_0^+}$ *is intrinsically ultracontractive with respect to* m_0,

(ii) $\forall\, t > 0\, \exists\, K_t \subseteq U, K_t$ *compact,* $\exists\, \varrho_{t,K_t} > 0 : P_t(\cdot, U) \le \varrho_{t,K_t} P_t(\cdot, K_t)$.

Proof "(i) $\Longrightarrow$ (ii):" Let $t > 0$ and choose some compact $K \subseteq U$. By Theorem 3.8 we have that

$$\forall x \in U : \frac{P_t(x,U)}{P_t(x,K)} \leq \frac{\alpha_t \int_K \varphi(x)\psi(y) m_0(\mathrm{d}y)}{\beta_t \int_U \varphi(x)\psi(y)\lambda_d(\mathrm{d}y)} \leq \frac{\alpha_t \int_K \psi(y) m_0(\mathrm{d}y)}{\beta_t \int_U \psi(y)\lambda_d(\mathrm{d}y)}.$$

"(ii) $\Longrightarrow$ (i):" Let $t > 0$. Then we infer from Lemma 3.9 that there exists some $\kappa_{t,K_t} > 0$ such that $p_t^{m_0}(x,\cdot) \leq \kappa_{t,K_t}\varphi(x)$ for all $x \in U$. Now the Chapman–Kolmogorov equation and the symmetry of $p_{t/2}^{m_0}$ yield that

$$\begin{aligned} p_t^{m_0}(x,y) &= \int_U p_{t/2}^{m_0}(x,z) p_{z/y}^{m_0} m_0(\mathrm{d}z) \\ &\leq \kappa_{t,K_t}\varphi(x) \int_U p_{t/2}^{m_0}(y,z) m_0(\mathrm{d}z) \\ &\leq \kappa_{t,K_t}^2 m_0(U)\varphi(x)\varphi(y) \end{aligned}$$

holds true for all $x, y \in U$. By means of Lemma 3.6 there exists some $\rho > 0$ such that $\varphi(y) \leq \rho\psi(y)$ for all $y \in U$, which results in

$$\forall x, y \in U : p_t^{m_0}(x,y) \leq \rho\kappa_{t,K_t}^2 m_0(U)\varphi(x)\psi(y).$$

□

By means of the previous theorem we obtain further equivalent characterisations of intrinsic ultracontractivity, which are given in the following theorem.

Theorem 3.11 *Assume that there exists a measure $m \in \mathcal{M}_f(U)$ which satisfies Hypothesis* **(H1')**. *Then the following properties are equivalent:*

(i) $(P_t)_{t\in\mathbb{R}_0^+}$ *is intrinsically ultracontractive with respect to m,*

(ii) $\forall\, s,t > 0 \,\exists\, \alpha_{s,t} \,\forall\, w,x,y,z \in U$:

$$\frac{p_t^m(x,y)}{p_t^m(x,z)} \leq \alpha_{s,t} \frac{p_s^m(w,y)}{p_s^m(w,z)}\,, \tag{3.11}$$

(iii) $\forall\, t > 0 \,\exists\, \alpha_t > 0 \,\forall\, w,x,y \in U$:

$$\frac{p_t^m(x,y)}{P_t(x,U)} \leq \alpha_t \frac{p_t^m(w,y)}{P_t(w,U)}\,. \tag{3.12}$$

Proof "(i) $\Longrightarrow$ (ii):"

Let $s, t > 0$, then by means of Theorem 3.8 $(P_t)_{t\in\mathbb{R}_0^+}$ being intrinsically ultracontractive implies that there exist $\alpha_s, \alpha_t, \beta_s, \beta_t > 0$ such that $\beta_s\varphi(x)\psi(y) \leq p_s^m(x,y) \leq \alpha_s\varphi(x)\psi(y)$, $\beta_t\varphi(x)\psi(y) \leq p_t^m(x,y) \leq \alpha_t\varphi(x)\psi(y)$ hold true for all $x, y \in U$. Thus, we infer that

$$\frac{p_s^m(x,y)p_t^m(w,z)}{p_s^m(x,z)p_t^m(w,y)} \leq \frac{\alpha_s\alpha_t\varphi(x)\psi(y)\varphi(w)\psi(z)}{\beta_s\beta_t\varphi(x)\psi(z)\varphi(w)\psi(y)} = \frac{\alpha_s\alpha_t}{\beta_s\beta_t}.$$

"(ii) $\Longrightarrow$ (iii):"

Let $s = t > 0$. Since $\alpha_t := \alpha_{s,t}$ in (3.11) does not depend on $z \in U$, we obtain that

$$\forall\, w, x, y \in U : p_t^m(x, y) \int_U p_t^m(w, z) m(\mathrm{d}z) \le \alpha_t p_t^m(w, y) \int_U p_t^m(x, z) m(\mathrm{d}z),$$

which is (3.12).

"(iii) $\Longrightarrow$ (i):" Fix some $t > 0$ as well as $x_0 \in U$, and let $\alpha_t > 0$ be such that

$$\frac{p_t^m(x_0, y)}{P_t(x_0, U)} \le \alpha_t \frac{p_t^m(\cdot, y)}{P_t(\cdot, U)}.$$

holds true for all $y \in U$. Hence

$$\int_K \frac{p_t^m(x_0, y)}{P_t(x_0, U)} m(\mathrm{d}y) \le \alpha_t \int_K \frac{p_t^m(\cdot, y)}{P_t(\cdot, U)} m(\mathrm{d}y),$$

which is equivalent to

$$\frac{P_t(\cdot, U)}{P_t(\cdot, K)} \le \alpha_t \frac{P_t(x_0, U)}{P_t(x_0, K)}.$$

With $\varrho_{t,K} := \alpha_t P_t(x_0, U) P_t(x_0, K)^{-1}$ this results in $P_t(\cdot, U) \le \varrho_{t,K} P_t(\cdot, K)$, and thus Theorem 3.10 yields the assertion. $\square$

Property (3.11) is referred to as **parabolic boundary Harnack principle**.

Now we present a sufficient condition for intrinsic ultracontractivity, which will be an indispensable tool in order to prove the main result of this thesis.

Theorem 3.12 *If Hypothesis* **(H1)** *is satisfied, then* $(P_t)_{t \in \mathbb{R}_0^+}$ *is intrinsically ultracontractive with respect to* m_0 *if for each* $t > 0$ *there exist some compact* $C_t \subseteq U$ *and a constant* $\kappa_{t,C_t} > 0$ *with*

$$\forall\, x \in C_t^{\complement} : P^x(\tau_{C_t^{\complement}} \le t | \tau_U > t) \ge \kappa_{t,C_t},$$

where $C_t^{\complement} := U \setminus C_t$.

Proof Fix some $t > 0$ and let $C_t \subseteq U$ and $\kappa_{t,C_t} > 0$ be such that C_t is compact and $P^x(\tau_{C_t^{\complement}} \le t, \tau_U > t) P_t(x, U)^{-1} \ge \kappa_{t,C_t}$ for all $x \in C_t^{\complement}$. In addition, choose some compact K_t with $C_t \subseteq K_t \subseteq U$ and $\inf\{\|x - y\|_2 : x \in \partial C_t, y \in \partial K_t\} > 0$.

Part I

We define the following $\mathscr{F}^X$–stopping times:

$$\begin{aligned} \sigma &:= \inf\{s \in \mathbb{R}_0^+ : X_s \in \partial K_t\}, \\ \zeta &:= \inf\{s \in \mathbb{R}_0^+ : X_s \in \partial C_t\}, \\ \upsilon &:= \inf\{s \in [\zeta, \infty) : X_s \in \partial K_t\} = \zeta + \sigma \circ \theta_\zeta, \end{aligned}$$

where θ_ζ denotes a shift operator as in Definition 1.9.

Observe that $P^x(t=\sigma) \le P^x(X_t \in \partial K_t) = 0$ for all $x \in U$, and thus $\mathbb{1}_{\{t\le\sigma\}}$ is continuous P^x–a.s. for every $x \in U$. Hence we deduce from Theorem 2.26 that $P^{(\cdot)}(t \le \sigma)$ is continuous on U. Similarly to the proof of Corollary 2.18 we infer from the Stroock-Varadhan Support Theorem (cf. Theorem 2.17) that $P^x(t \le \sigma) > 0$ for all $x \in \partial C_t$. Furthermore, $X_\zeta(\omega) \in \partial C_t$ for all $\omega \in \{\zeta < \infty\}$, and moreover ∂C_t is compact. Therefore, we conclude that

$$\eta_{t,C_t} := \inf_{\omega\in\{\zeta<\infty\}} P^{X_\zeta(\omega)}(t \le \sigma) \ge \min_{x\in\partial C_t} P^x(t\le\sigma) > 0.$$

Now we deduce from the strong Markov property that

$$\begin{aligned}
P_t(x,K_t)P_t(x,U)^{-1} &\ge P^x(t\in[\zeta,\upsilon])P_t(x,U)^{-1}\\
&= P^x(\{\zeta\le t\}\cap\{t\le\upsilon\})P_t(x,U)^{-1}\\
&= E_x\left(\mathbb{1}_{\{\zeta\le t\}}P^x(t-\zeta\le\sigma\circ\theta_\zeta|\mathscr{F}_\zeta)\right)P_t(x,U)^{-1}\\
&\ge E_x\left(\mathbb{1}_{\{\zeta\le t\}}P^{X_\zeta}(t\le\sigma)\right)P_t(x,U)^{-1}\\
&\ge \eta_{t,C_t}P^x(\zeta\le t)P_t(x,U)^{-1}\\
&\ge \eta_{t,C_t}\kappa_{t,C_t}
\end{aligned}$$

holds for all $x \in C_t^{\complement}$, since $P^x(\zeta \le t)P_t(x,U)^{-1} = P^x(\tau_{C_t^{\complement}} \le t, \tau_U > t)P_t(x,U)^{-1}$ holds true for any $x \in C_t^{\complement}$. With $\alpha^I_{t,C_t} := \eta^{-1}_{t,C_t}\kappa^{-1}_{t,C_t} > 0$ this results in

$$P_t(x,U) \le \alpha^I_{t,C_t} P_t(x,K_t)$$

for all $x \in C_t^{\complement}$.

Part II

The continuity and positivity of $p_t(\cdot,\cdot)$, $t>0$, implies that $P_t(\cdot,K_t)$ is continuous and positive, and hence $\min_{x\in C_t} P_t(x,K_t) > 0$. Thus, for any $t>0$ there exists some $\alpha^{II}_{t,C_t} > 0$ such that

$$P_t(x,U) \le \max_{y\in C_t} P_t(y,U) \le \alpha^{II}_{t,C_t} \min_{y\in C_t} P_t(y,K_t) \le \alpha^{II}_{t,C_t} P_t(x,K_t)$$

holds true for all $x \in C_t$.

Part III

For each $t>0$ set $\alpha_{t,C_t} := \max(\alpha^I_{t,C_t}, \alpha^{II}_{t,C_t})$. Then we infer from Part I and Part II that

$$\forall\, t>0 : P_t(\cdot,U) \le \alpha_{t,C_t} P_t(\cdot,K_t),$$

and hence the assertion follows from Theorem 3.10. □

Note that we have shown the above characterisations of intrinsic ultracontractivity under the assumption that Hypothesis **(H1)** is satisfied. In the following we will develop a tool which enables us to deal with the case that Hypothesis **(H2)** is satisfied. To this end we will consider the formal adjoint of T, which is defined as follows:

Definition 3.13 *If Hypothesis* **(H2)** *is satisfied, then the operator* $T^* : C_K^2(U) \to C_K(U)$*, defined by*

$$T^* f = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij} f) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i f)$$

for all $f \in C_K^2(U)$*, is called the* **formal adjoint** *of* T*.*

Lemma 3.14 *Presume that Hypothesis* **(H2)** *is satisfied. Then* T^* *is the restriction to* $C_K^2(U)$ *of the generator of the formal adjoint semigroup* $(T_t^*)_{t\in\mathbb{R}_0^+}$*, which in fact motivates the name for that semigroup.*

Proof Our procedure to prove the assertion is to give a characterisation of the generator of $(T_t^*)_{t\in\mathbb{R}_0^+}$, and to show that this characterisation uniquely determines the generator on $C_K^2(U)$. Then we will show that T^* satisfies this characterisation, and thus we will conclude that T^* is the restriction of the generator of $(T_t^*)_{t\in\mathbb{R}_0^+}$ to $C_K^2(U)$.

Part I

We deduce from (3.5) that

$$\forall\, f, g \in C_K^2(U)\, \forall\, t > 0 : \int_U f(x) t^{-1}(T_t^* - \mathrm{id}\,) g(x) \lambda_d(\mathrm{d}x) = \int_U g(x) t^{-1}(T_t - \mathrm{id}\,) f(x) \lambda_d(\mathrm{d}x).$$

Let T' denote the generator of $(T_t^*)_{t\in\mathbb{R}_0^+}$. Then the aforementioned equality results in

$$\begin{aligned} \int_U f(x) T' g(x) \lambda_d(\mathrm{d}x) &= \lim_{n\to\infty} \int_U f(x) n(T_{1/n}^* - \mathrm{id}\,) g(x) \lambda_d(\mathrm{d}x) \\ &= \lim_{n\to\infty} \int_U g(x) n(T_{1/n} - \mathrm{id}\,) f(x) \lambda_d(\mathrm{d}x) \qquad (3.13) \\ &= \int_U g(x) T f(x) \lambda_d(\mathrm{d}x) \end{aligned}$$

for all $f, g \in C_K^2(U)$.

Let $B \subseteq U$ be compact. According to Lemma B.21 there exists a monotonically decreasing sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_K^2(U)$ converging pointwise to $\mathbb{1}_B$ as $n \to \infty$, and thus we obtain by the Dominated Convergence Theorem that

$$\int_B T' g(x) \lambda_d(\mathrm{d}x) = \lim_{n\to\infty} \int_U f_n(x) T' g(x) \lambda_d(\mathrm{d}x) \overset{(3.13)}{=} \lim_{n\to\infty} \int_U g(x) T f_n(x) \lambda_d(\mathrm{d}x)$$

holds true for all $g \in C_K^2(U)$. Since $T'g$ is continuous, this shows that T' is uniquely defined on $C_K^2(U)$ by (3.13).

Part II

In view of Part I we have to show that $\int_U gTf \;\mathrm{d}\lambda_d = \int_U fT^*g \;\mathrm{d}\lambda_d$ holds for all $f, g \in C_K^2(U)$. By the d–dimensional integration by parts formula we obtain that

$$\int_U gTf \;\mathrm{d}\lambda_d \;=\; \int_U \frac{1}{2} \sum_{i,j=1}^{d} g a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f \;\mathrm{d}\lambda_d + \int_U \sum_{i=1}^{d} g b_i \frac{\partial}{\partial x_i} f \;\mathrm{d}\lambda_d$$

$$
\begin{aligned}
&= -\int_U \frac{1}{2}\sum_{i,j=1}^{d}\left[\frac{\partial}{\partial x_i}(g a_{ij})\right]\left[\frac{\partial}{\partial x_j} f\right]\,\mathrm{d}\lambda_d - \int_U \sum_{i=1}^{d} f\frac{\partial}{\partial x_i}(g b_i)\,\mathrm{d}\lambda_d \\
&= \int_U \frac{1}{2}\sum_{i,j=1}^{d} f\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij} g)\,\mathrm{d}\lambda_d - \int_U \sum_{i=1}^{d} f\frac{\partial}{\partial x_i}(b_i g)\,\mathrm{d}\lambda_d \\
&= \int_U f T^* g\,\mathrm{d}\lambda_d
\end{aligned}
$$

holds true for all $f, g \in C_K^2(U)$, which proves the assertion. Note that the d–dimensional integration by parts formula requires a domain with piecewise smooth boundary. However, because $f, g \in C_K^2(U)$, we can consider a sub–domain $U' \subseteq U$ with piecewise smooth boundary and such that $\operatorname{supp}(f) \cup \operatorname{supp}(g) \subseteq U'$. Hence we can apply the d–dimensional integration by parts formula, and thus we deduce that $T^* = T'$, i.e., T^* is the restriction to $C_K^2(U)$ of the generator of the formal adjoint semigroup $(T_t^*)_{t\in\mathbb{R}_0^+}$. □

Lemma 3.15 *Presume that Hypothesis* **(H2)** *is satisfied, and let $t > 0$. Then there exists a function $p_t^* \in C_b(U \times U, \mathbb{R}^+)$ satisfying $T_t^* f = \int_U p_t^*(\cdot, y) f(y)\lambda_d(dy)$ for every $f \in C_b(U)$, and such that $p_t(x, y) = p_t^*(y, x)$ for all $x, y \in U$.*

Proof At first observe that

$$
\begin{aligned}
&T^* f \\
&= \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij} f) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i f) \\
&= \frac{1}{2}\sum_{i,j=1}^{d}\left(a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} f + 2\left[\frac{\partial}{\partial x_i} a_{ij}\right]\left[\frac{\partial}{\partial x_j} f\right] + f\frac{\partial^2}{\partial x_i\partial x_j} a_{ij}\right) - \sum_{i=1}^{d}\left(b_i\frac{\partial}{\partial x_i} f + f\frac{\partial}{\partial x_i} b_i\right) \\
&= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} f - \sum_{i=1}^{d} b_i\frac{\partial}{\partial x_i} f + \sum_{i,j=1}^{d}\left[\frac{\partial}{\partial x_j} a_{ij}\right]\left[\frac{\partial}{\partial x_i} f\right] \qquad (3.14) \\
&\quad - \sum_{i=1}^{d} f\frac{\partial}{\partial x_i} b_i + \frac{1}{2}\sum_{i,j=1}^{d} f\frac{\partial^2}{\partial x_i\partial x_j} a_{ij} \\
&= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} f - \sum_{i=1}^{d}\left(b_i - \sum_{j=1}^{d}\frac{\partial}{\partial x_j} a_{ij}\right)\frac{\partial}{\partial x_i} f - \sum_{i=1}^{d} f\frac{\partial}{\partial x_i}\left(b_i - \frac{1}{2}\sum_{j=1}^{d}\frac{\partial}{\partial x_j} a_{ij}\right)
\end{aligned}
$$

holds true for all $f \in C_K^2(U)$. Hence, Theorem 0.6 in §6 of the appendix of [Dyn65II] yields that there exists a function $p_t^* \in C_b(U \times U, \mathbb{R}_0^+)$ satisfying $T_t^* f = \int_U p_t^*(\cdot, y) f(y)\lambda_d(\mathrm{d}y)$ for every $f \in C_b(U)$. Let $B_1, B_2 \in \mathscr{B}(U)$ be compact. By Lemma B.21 there exist monotonically decreasing sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subseteq C_K^2(U)$ converging pointwise to $\mathbb{1}_{B_1}$ and $\mathbb{1}_{B_2}$, respectively, as $n \to \infty$. In conjunction with (3.5) and the Dominated Convergence Theorem our considerations above result in

$$
\int_{B_1}\int_{B_2} p_t(x, y)\lambda_d(\mathrm{d}y)\lambda_d(\mathrm{d}x) = \lim_{m\to\infty}\lim_{n\to\infty}\int_U f_n(x) T_t g_m(x)\lambda_d(\mathrm{d}x)
$$

$$\begin{aligned}
&= \lim_{m\to\infty}\lim_{n\to\infty}\int_U g_m(y)T_t^* f_n(y)\lambda_d(\mathrm{d}y)\\
&= \int_{B_2}\int_{B_1} p_t^*(y,x)\lambda_d(\mathrm{d}x)\lambda_d(\mathrm{d}y)\\
&= \int_{B_1}\int_{B_2} p_t^*(y,x)\lambda_d(\mathrm{d}y)\lambda_d(\mathrm{d}x).
\end{aligned}$$

Therefore, we conclude that $p_t(x,y) = p_t^*(y,x)$ for all $x, y \in U$. □

In particular, we had shown in the proof of the previous lemma that if Hypothesis **(H2)** is satisfied, then

$$T^* f = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} f - \sum_{i=1}^{d}\left(b_i - \sum_{j=1}^{d}\frac{\partial}{\partial x_j}a_{ij}\right)\frac{\partial}{\partial x_i} f - \sum_{i=1}^{d} f\frac{\partial}{\partial x_i}\left(b_i - \frac{1}{2}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}a_{ij}\right)$$

holds for all $f \in C_K^2(U)$. If Hypothesis **(H2)** is satisfied, then in view of the previous equation we consider a linear operator $\tilde{T} : C_K^2(U) \to C_K(U)$ defined by

$$\forall\, f \in C_K^2(U) : \tilde{T} f = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} f - \sum_{i=1}^{d}\left(b_i - \sum_{j=1}^{d}\frac{\partial}{\partial x_j}a_{ij}\right)\frac{\partial}{\partial x_i} f.$$

Since by Hypothesis **(H2)** the coefficients of $\tilde{T}$ satisfy all the assumptions which we have made in Section 3.1, we can follow the lines of our procedure in that section in order to obtain that for any $x \in U$ the corresponding martingale problem has a unique solution $\tilde{P}^x$ on $(\Omega, \mathscr{F})$. Furthermore, for any $t > 0$ the measure $\tilde{P}_t(x,\cdot)$ possesses a $\lambda_d|_U$–density $\tilde{p}_t(x,\cdot)$, with properties as in Theorem 3.1, where $(\tilde{P}_t)_{t\in\mathbb{R}_0^+}$ denotes the sub–Markov semigroup of transition kernels associated with $\tilde{P}^x$. Moreover, as in Theorem 3.3 we deduce that there exists a positive $\tilde{\gamma}$ and a positive $\tilde{\varphi} \in C_0(U)$ such that $e^{-\tilde{\gamma}t}\tilde{\varphi}(x) = \int_U \tilde{P}_t(x,\mathrm{d}y)\tilde{\varphi}(y)$ for all $x \in U$.

Definition 3.16 *Let $g \in C_b(U)$. Then we define a family $(T_t^g)_{t\in\mathbb{R}_0^+}$ of linear operators $T_t^g : C_0(U) \to C_0(U)$ by*

$$\forall\, t > 0\,\forall\, f \in C_0(U)\,\forall\, x \in U : T_t^g f(x) = E_x\left(\exp\left[\int_{[0,t]} g\circ X_s \lambda(ds)\right] f\circ X_t\right).$$

According to Theorem 1 in Section 17 of Chapter 2 in [Itô04] we have that $(T_t^g)_{t\in\mathbb{R}_0^+}$ is a semigroup, which we call the **Kac semigroup** *with rate function g associated with $(T_t)_{t\in\mathbb{R}_0^+}$. Furthermore, we denote by T^g the generator of $(T_t^g)_{t\in\mathbb{R}_0^+}$.*

Lemma 3.17 *Let $g \in C_b(U)$. Then $C_K^2(U) \subseteq \mathcal{D}_{T^g}$, where $\mathcal{D}_{T^g}$ denotes the domain of T^g. In addition, we have that $T^g f = Tf + g\cdot f$ for all $f \in C_K^2(U)$, i.e., $T^g f(x) = Tf(x) + g(x)f(x)$ for all $x \in U$.*

Proof Confer Theorem 3 in Section 17 of Chapter 2 in [Itô04]. □

Theorem 3.18 *Presume that Hypothesis* **(H2)** *is satisfied. Let $t > 0$ and assume that*

$$\exists\, K_t \subseteq U, K_t \text{ compact}, \exists\, \alpha_{t,K_t} > 0 : \tilde{P}_t(\cdot, U) \leq \alpha_{t,K_t} \tilde{P}_t(\cdot, K_t).$$

Then there exists some ρ_{t,K_t} such that $p_t(\cdot, y) \leq \rho_{t,K_t} \psi(y)$ holds for all $y \in U$.

Proof

Part I

Choose some compact $K_t \subseteq U$ and $\kappa_{t,K_t} > 0$ with

$$\forall\, x \in U : \tilde{p}_t(x, \cdot) \leq \kappa_{t,K_t} \tilde{\varphi}(x). \tag{3.15}$$

Note that such K_t and κ_{t,K_t} exist by an argument as in Lemma 3.9. Moreover, as in Lemma 3.6 we deduce from Proposition 3 in [GQZ88] that there exist some $\tilde{\alpha}, \beta > 0$ such that

$$\beta \delta(x) \leq \psi(x) \quad \text{and} \quad \tilde{\varphi}(x) \leq \tilde{\alpha} \delta(x) \tag{3.16}$$

hold for all $x \in U$.

Here we have utilised that by Hypothesis **(H2)** the coefficients of $\tilde{T}$ satisfy all the assumptions which we have made in Section 3.1. In particular, this ensures that an argument as in Lemma 3.9 is applicable, and that the assumptions of Proposition 3 in [GQZ88] are satisfied if we consider $\tilde{T}$.

Part II

Choose some $c_0 \in \mathbb{R}^+$ with

$$c_0 \geq -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(b_i - \frac{1}{2} \sum_{j=1}^{d} \frac{\partial}{\partial x_j} a_{ij} \right),$$

which exists since $\frac{\partial^2}{\partial x_i \partial x_j} a_{ij}$ and $\frac{\partial}{\partial x_i} b_i$ are bounded for all $i, j \in \{1, \dots, d\}$. Furthermore, put

$$c := \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(b_i - \frac{1}{2} \sum_{j=1}^{d} \frac{\partial}{\partial x_j} a_{ij} \right) + c_0 \geq 0.$$

Consider the Kac semigroup $(T_s^{c_0})_{s \in \mathbb{R}_0^+}$ with rate function $g :\equiv -c_0$ associated with $(T_s)_{s \in \mathbb{R}_0^+}$. Then we infer that

$$\begin{aligned} \int_U p_t^{c_0}(x, y) f(y) \lambda_d(\mathrm{d}y) &= T_t^{c_0} f(x) \\ &= \exp\left[-\int_{[0,t]} c_0 \lambda(\mathrm{d}s) \right] \int_U P_t(x, \mathrm{d}y) f(y) \\ &= e^{-c_0 t} \int_U p_t(x, y) f(y) \lambda_d(\mathrm{d}y) \end{aligned}$$

holds true for all $x \in U$ and every $f \in C_0(U)$. Let $B \subseteq U$ be compact, and in the light of Lemma B.21 let $(f_n)_{n\in\mathbb{N}}$ be a monotonically decreasing sequence in $C_0(U)$, which converges pointwise to $\mathbb{1}_B$ as $n \to \infty$. Then the Dominated Convergence Theorem yields that

$$\begin{aligned}\forall\, t > 0\, \forall\, x \in U : \int_B p_t^{c_0}(x,y)\lambda_d(\mathrm{d}y) &= \lim_{n\to\infty} \int_U p_t^{c_0}(x,y) f_n(y)\lambda_d(\mathrm{d}y) \\ &= e^{-c_0 t} \lim_{n\to\infty} \int_U p_t(x,y) f_n(y)\lambda_d(\mathrm{d}y) \\ &= e^{-c_0 t} \int_B p_t(x,y)\lambda_d(\mathrm{d}y),\end{aligned}$$

which shows that $p_t^{c_0}(x,\cdot) = e^{-c_0 t} p_t(x,\cdot)$ for all $x \in U$. Denote the generator of $(T_s^{c_0})_{s\in\mathbb{R}_0^+}$ by T^{c_0}. Analogously to (3.14) we obtain that T^{c_0*}, the formal adjoint of T^{c_0}, is given on $C_K^2(U)$ by

$$\forall\, f \in C_K^2(U) : T^{c_0*} f = \frac{1}{2}\sum_{i,j=1}^d a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f - \sum_{i=1}^d \left(b_i - \sum_{j=1}^d \frac{\partial}{\partial x_j} a_{ij} \right) \frac{\partial}{\partial x_i} f - cf.$$

We infer from Theorem 0.6 in §6 of the appendix of [Dyn65II] that there exists a function $p_t^{c_0*} \in C_b(U \times U, \mathbb{R}_0^+)$ satisfying $T_t^{c_0*} f = \int_U p_t^{c_0*}(\cdot,y) f(y)\lambda_d(\mathrm{d}y)$ for every $f \in C_b(U)$, and as in the proof of Lemma 3.15 we deduce that

$$\forall\, t > 0\, \forall\, x \in U : p_t^{c_0*}(x,\cdot) = p_t^{c_0}(\cdot,x) = e^{-c_0 t} p_t(\cdot,x). \tag{3.17}$$

Part III

Now observe that $(T_s^{c_0*})_{s\in\mathbb{R}_0^+}$ is the Kac semigroup with rate function $-c$ associated with $(\tilde{T}_s)_{s\in\mathbb{R}_0^+}$. Therefore, we conclude that

$$\begin{aligned}\int_U p_t^{c_0*}(x,y) f(y)\lambda_d(\mathrm{d}y) &= T_t^{c_0*} f(x) = \int_U \exp\left(-\int_{[0,t]} c \circ X_s \lambda(\mathrm{d}s) \right) f \circ X_t \tilde{P}(x,\cdot) \\ &\leq \int_U \tilde{p}_t(x,y) f(y)\lambda_d(\mathrm{d}y)\end{aligned}$$

holds for any $x \in U$ and all $f \in C_0$. Analogously to above we infer that $p_t^{c_0*}(x,\cdot) \leq \tilde{p}_t(x,\cdot)$ holds for all $x \in U$ and $t > 0$, which by means of (3.15), (3.16) and (3.17) results in

$$p_t(\cdot,x) = e^{c_0 t} p_t^{c_0*}(x,\cdot) \leq e^{c_0 t}\tilde{p}_t(x,\cdot) \leq \kappa_{t,K_t} e^{c_0 t} \tilde{\varphi}(x) \kappa_{t,K_t} \tilde{\alpha} e^{c_0 t} \delta(x) \leq \rho_{t,K_t} \beta \delta(x) \leq \rho_{t,K_t} \psi(x),$$

where $\rho_{t,K_t} := \kappa_{t,K_t} \tilde{\alpha} e^{c_0 t} \beta^{-1}$. □

Now we are in a position to prove similar results as in Theorem 3.10, Theorem 3.11 and Theorem 3.12 in the situation that Hypothesis **(H2)** is satisfied.

Theorem 3.19 *Presume that Hypothesis* **(H2)** *is satisfied, then* $(P_t)_{t\in\mathbb{R}_0^+}$ *is intrinsically ultracontractive if for every* $t > 0$ *there exist some compact* $K_t \subseteq U$ *and some* $\varrho_{t,K_t} > 0$ *with*

$$P_t(\cdot,U) \leq \varrho_{t,K_t} P_t(\cdot,K_t) \quad \textit{and} \quad \tilde{P}_t(\cdot,U) \leq \varrho_{t,K_t} \tilde{P}_t(\cdot,K_t). \tag{3.18}$$

Proof Let $t > 0$. We deduce from Lemma 3.9 and Theorem 3.18 in conjunction with the Chapman–Kolmogorov equation that there exist some positive constants κ_{t,K_t} and ρ_{t,K_t} with

$$\forall\, x, y \in U\, \forall\, t > 0 : p_t(x,y) = \int_U p_{t/2}(x,z) p_{t/2}(z,y) \lambda_d(\mathrm{d}z) \leq \kappa_{t,K_t} \rho_{t,K_t} \lambda_d(U) \varphi(x) \psi(y).$$

□

Analogously to the proofs of Theorem 3.10 one can show that (3.18) holds true if Hypothesis **(H2)** is satisfied and if $(P_t)_{t\in\mathbb{R}_0^+}$ and $(\tilde{P}_t)_{t\in\mathbb{R}_0^+}$ are intrinsically ultracontractive.

Theorem 3.20 *Presume that Hypothesis* **(H2)** *is satisfied and consider*

(i) $(P_t)_{t\in\mathbb{R}_0^+}$ *and* $(\tilde{P}_t)_{t\in\mathbb{R}_0^+}$ *are intrinsically ultracontractive,*

(ii) $\forall\, s, t > 0\, \exists\, \alpha_{s,t}\, \forall\, w, x, y, z \in U$:

$$\frac{p_t(x,y)}{p_t(x,z)} \leq \alpha_{s,t} \frac{p_s(w,y)}{p_s(w,z)} \quad \textit{and} \quad \frac{\tilde{p}_t(x,y)}{\tilde{p}_t(x,z)} \leq \alpha_{s,t} \frac{\tilde{p}_s(w,y)}{\tilde{p}_s(w,z)}\,, \tag{3.19}$$

(iii) $\forall\, t > 0\, \exists\, \alpha_t > 0\, \forall\, w, x, y \in U$:

$$\frac{p_t(x,y)}{P_t(x,U)} \leq \alpha_t \frac{p_t(w,y)}{P_t(w,U)} \quad \textit{and} \quad \frac{\tilde{p}_t(x,y)}{\tilde{P}_t(x,U)} \leq \alpha_t \frac{\tilde{p}_t(w,y)}{\tilde{P}_t(w,U)}\,, \tag{3.20}$$

(iv) $(P_t)_{t\in\mathbb{R}_0^+}$ *is intrinsically ultracontractive.*

Then the following implications hold true: (*i*) $\Longrightarrow$ (*ii*) $\Longrightarrow$ (*iii*) $\Longrightarrow$ (*iv*).

Proof We can prove these assertions along the lines of the proof of Theorem 3.11 by applying the methods of that proof to both semigroups $(P_t)_{t\in\mathbb{R}_0^+}$ and $(\tilde{P}_t)_{t\in\mathbb{R}_0^+}$, and by refering to Theorem 3.19 instead of Theorem 3.10. □

Remark 3.21 Note that in contrast to Theorem 3.10 and Theorem 3.11 we do not obtain equivalent characterisations of intrinsic ultracontractivity if "only" Hypothesis **(H2)** is satisfied, since we cannot show that (3.18) or (3.19) or (3.20) implies that $(\tilde{P}_t)_{t\in\mathbb{R}_0^+}$ is intrinsically ultracontractive. ◊

Theorem 3.22 *If Hypothesis* **(H2)** *is satisfied, then* $(P_t)_{t\in\mathbb{R}_0^+}$ *is intrinsically ultracontractive if for each* $t > 0$ *there exist some compact* $C_t \subseteq U$ *and a constant* $\kappa_{t,C_t} > 0$ *such that*

$$P^x(\tau_{C_t^{\complement}} \leq t | \tau_U > t) \geq \kappa_{t,C_t} \quad \textit{and} \quad \tilde{P}^x(\tau_{C_t^{\complement}} \leq t | \tau_U > t) \geq \kappa_{t,C_t}$$

hold for all $x \in C_t^{\complement}$, *where* $C_t^{\complement} := U \setminus C_t$.

Proof We can prove the assertion on the lines of the proof of Theorem 3.12 by applying the methods developed in that proof to both semigroups $(P_t)_{t\in\mathbb{R}_0^+}$ and $(\tilde{P}_t)_{t\in\mathbb{R}_0^+}$, and by refering to Theorem 3.19 instead of Theorem 3.10. □

Definition 3.23 *If* $\int_{[0,\infty]} p_t(x,y)\lambda(dt) < \infty$ *for all* $x, y \in U$*, then we call the function* $G : U \times U \to \mathbb{R}_0^+$*, defined by*

$$\forall\, x, y \in U : G(x,y) = \int_{[0,\infty]} p_t(x,y)\lambda(dt),$$

the **Green's function** *for* T *on* U*.*

Before we can proof our main theorem we will need some more auxiliary results, which we will provide in the following lemmas.

Lemma 3.24 *Let* $\alpha > 0$*. Then there exists some* $\varepsilon > 0$ *such that*

$$\int_{B(x,\varepsilon)} \|x-y\|_2^{1-d}\lambda_d(dy) \le \alpha$$

for all $x \in \mathbb{R}^d$*.*

Proof Let $x \in \mathbb{R}^d$. We describe $y \in B(0,\varepsilon)$ by d–dimensional spherical polar coordinates, i.e.,

$$\begin{aligned}
y_1 &= r\cos\theta_1\\
y_2 &= r\sin\theta_1\cos\theta_2\\
&\vdots\\
y_{d-1} &= r\sin\theta_1\ldots\sin\theta_{d-2}\cos\theta_{d-1}\\
y_d &= r\sin\theta_1\ldots\sin\theta_{d-2}\sin\theta_{d-1},
\end{aligned}$$

where $\theta_1,\ldots,\theta_{d-2} \in [0,2\pi]$, $\theta_{d-1} \in [0,\pi]$ and $r = \|y\|_2 \in [0,\varepsilon)$. This results in

$$\begin{aligned}
&\int_{B(x,\varepsilon)} \|x-y\|_2^{1-d}\lambda_d(\mathrm{d}y)\\
=\;& \int_{B(0,\varepsilon)} \|y\|_2^{1-d}\lambda_d(\mathrm{d}y)\\
\overset{(*)}{=}\;& \int_{[0,\pi]}\cdots\int_{[0,2\pi]}\int_{[0,\varepsilon)} \frac{r^{d-1}}{\|y\|_2^{d-1}}\sin^{d-2}\theta_1\ldots\sin\theta_{d-2}\lambda(\mathrm{d}r)\lambda(\mathrm{d}\theta_1)\ldots\lambda(\mathrm{d}\theta_{d-1})\\
=\;& \varepsilon\pi\int_{[0,2\pi]}\ldots,\int_{[0,2\pi]}\sin^{d-2}\theta_1\ldots\sin\theta_{d-2}\lambda(\mathrm{d}\theta_1)\ldots\lambda(\mathrm{d}\theta_{d-2}),
\end{aligned}$$

which yields the assertion with

$$\varepsilon \le \alpha\pi^{-1}\left[\int_{[0,2\pi]}\ldots,\int_{[0,2\pi]}\sin^{d-2}\theta_1\ldots\sin\theta_{d-2}\lambda(\mathrm{d}\theta_1)\ldots\lambda(\mathrm{d}\theta_{d-2})\right]^{-1}.$$

Note that above we have used the transformation formula in order to obtain $(*)$. □

Lemma 3.25 *Let $\alpha > 0$. Then there exists some compact $K \subseteq U$ with*

$$\forall\, x \in U : \int_{U \setminus K} \|x-y\|_2^{1-d} \lambda_d(dy) \leq \alpha.$$

Proof In view of Lemma 3.24 we choose some $\varepsilon > 0$ with $\int_{B(x,\varepsilon)} \|x-y\|_2^{1-d} \lambda_d(\mathrm{d}y) \leq 2^{-1}\alpha$ for all $x \in U$. In addition, for any $x \in U$ we denote the complement of $B(x,\varepsilon)$ by $U^{\complement}_{\|\cdot\|_2}(x,\varepsilon)$. Then we infer that

$$\begin{aligned} & \int_{U \setminus K} \|x-y\|_2^{1-d} \lambda_d(\mathrm{d}y) \\ = & \int_{(U\setminus K)\cap B(x,\varepsilon)} \|x-y\|_2^{1-d} \lambda_d(\mathrm{d}y) + \int_{(U\setminus K)\cap U^{\complement}_{\|\cdot\|_2}(x,\varepsilon)} \|x-y\|_2^{1-d} \lambda_d(\mathrm{d}y) \\ \leq & \frac{\alpha}{2} + \varepsilon^{1-d} \lambda_d(U \setminus K) \end{aligned}$$

holds true for all $K \subseteq U$ and each $x \in U$. Note that $\|x-y\|^{1-d} \leq \varepsilon^{1-d}$ for all $y \in U^{\complement}_{\|\cdot\|_2}(x,\varepsilon)$, because $1-d < 0$. By means of the regularity of λ_d we can choose some compact $K \subseteq U$ with $\lambda_d(U \setminus K) \leq 2^{-1}\alpha\varepsilon^{d-1}$, which yields the assertion. In conclusion we would like to emphasise that in our considerations above the radius ε of $B(x,\varepsilon)$ does not depend on $x \in U$. Therefore, also the choice of K is independent of x. □

Lemma 3.26 *Let $t_0 > 0$, then there exists some ρ_{t_0} such that*

$$P^x(\tau_U > t) \geq \rho_{t_0} e^{-\gamma t} \delta(x)$$

holds true for all $x \in U$ and any $t > t_0$. If Hypothesis **(H2)** *is satisfied, then there also exists a $\tilde{\rho}_{t_0}$ with*

$$\forall\, x \in U\, \forall\, t > t_0 : \tilde{P}^x(\tau_U > t) \geq \tilde{\rho}_{t_0} e^{-\gamma t} \delta(x).$$

Proof Confer (15) in Theorem 2 in [GQZ88]. □

For the time being, fix some $t > 0$. We infer from (3.3) Theorem (ii) in [GW82] in conjunction with the theorem in [HS82] that the Green's function G for T on U satisfies

$$G(x,y) \leq \kappa_t \delta(x) \|x-y\|_2^{1-d} \tag{3.21}$$

for some $\kappa_t > 0$.

In view of Lemma 3.26 choose some ρ_t such that $P^x(\tau_U > t) \geq \rho_t e^{-\gamma t}\delta(x)$ for all $x \in U$, and furthermore let $\alpha_t \leq 2^{-1} t \kappa_t^{-1} \rho_t e^{-\gamma t}$. In the light of Lemma 3.25 choose a compact $C_t \subseteq U$ with

$$\forall\, x \in C_t^{\complement} : \int_{C_t^{\complement}} \|x-y\|_2^{1-d} \lambda_d(\mathrm{d}y) \leq \alpha_t, \tag{3.22}$$

where $C_t^{\complement} := U \setminus C_t$

Applying the set–up just established, we obtain the following lemma:

Lemma 3.27 *Let $x \in C_t^{\complement}$ and $t > 0$. Then*

$$E_x(\tau_{C_t^{\complement}}) \leq \alpha_t \kappa_t \delta(x),$$

where $\tau_{C_t^{\complement}} := \inf\{t \in \mathbb{R}_0^+ : X_t \notin C_t^{\complement}\}$. If Hypothesis **(H2)** *is satisfied, then we also have $\tilde{E}_x(\tau_{C_t^{\complement}}) \leq \tilde{\alpha}_t \tilde{\kappa}_t \delta(x)$, where $\tilde{E}_x$ denotes the expectation with respect to $\tilde{P}^x$, and where $\tilde{\alpha}_t$ and $\tilde{\kappa}_t$ are defined analogously to α_t and κ_t.*

Proof We have

$$\begin{aligned}
E_x(\tau_{C_t^{\complement}}) &= \int_\Omega \tau_{C_t^{\complement}} \, \mathrm{d}P^x = \int_{[0,\infty)} t P^x_{\tau_{C_t^{\complement}}}(\mathrm{d}t) = \int_{[0,\infty)} P^x(\tau_{C_t^{\complement}} > t)\lambda(\mathrm{d}t) \\
&\leq \int_{[0,\infty)} P_t(x, C_t^{\complement})\lambda(\mathrm{d}t) = \int_{[0,\infty)} \int_{C_t^{\complement}} p_t(x,y)\lambda_d(\mathrm{d}y)\lambda(\mathrm{d}t) \\
&= \int_{C_t^{\complement}} G(x,y)\lambda_d(\mathrm{d}y) \overset{(3.21)}{\leq} \kappa_t \delta(x) \int_{C_t^{\complement}} \|x-y\|_2^{1-d} \lambda_d(\mathrm{d}y) \\
&\overset{(3.22)}{\leq} \alpha_t \kappa_t \delta(x).
\end{aligned}$$

On the lines of our above considerations we obtain $\tilde{E}_x(\tau_{C_t^{\complement}}) \leq \tilde{\alpha}_t \tilde{\kappa}_t \delta(x)$ if Hypothesis **(H2)** is satisfied. □

Now we are ready to prove the main result of this thesis, which states that Hypothesis **(H1)** or Hypothesis **(H2)** is sufficient in order that $(P_t)_{t \in \mathbb{R}_0^+}$ is intrinsically ultracontractive with respect to some inner regular finite measure on $(U, \mathscr{B}(U))$.

Theorem 3.28 *If Hypothesis* **(H1)** *is satisfied, then $(P_t)_{t \in \mathbb{R}_0^+}$ is intrinsically ultracontractive on U with respect to m_0, and if Hypothesis* **(H2)** *is satisfied, then $(P_t)_{t \in \mathbb{R}_0^+}$ is intrinsically ultracontractive on U [with respect to λ_d].*

Proof Let $t > 0$. By Markov's inequality and Lemma 3.27 we obtain that

$$P^x(\tau_{C_t^{\complement}} > t) \leq t^{-1} E_x(\tau_{C_t^{\complement}}) \leq 2^{-1} \rho_t e^{-\gamma t}$$

for all $x \in C_t^{\complement}$. Since $P^x(\tau_U > t) \geq \rho_t e^{-\gamma t}\delta(x)$ for all $x \in U$, we infer that

$$\frac{P^x(\tau_{C_t^{\complement}} \leq t, \tau_U > t)}{P^x(\tau_U > t)} = 1 - \frac{P^x(\tau_{C_t^{\complement}} > t, \tau_U > t)}{P^x(\tau_U > t)} = 1 - \frac{P^x(\tau_{C_t^{\complement}} > t)}{P^x(\tau_U > t)} \geq \frac{1}{2}$$

holds true for all $x \in C_t^{\complement}$. If Hypothesis **(H2)** is satisfied, then the same method as above yields that $\tilde{P}^x(\tau_{C_t^{\complement}} \leq t, \tau_U > t)\tilde{P}^x(\tau_U > t)^{-1} \geq 2^{-1}$ holds true for all $x \in C_t^{\complement}$. By means of Theorem 3.12 and Theorem 3.22 this yields the assertion. □

Now we discuss briefly the differences between the main result in [KS06a] and our main result. Kim and Song consider a bounded Lipschitz domain $D \subseteq \mathbb{R}^d$ and an operator $L : C_K^2(U) \to C_K(U)$ whose principal part is in divergence form, i.e.,

$$\forall\, f \in C_K^2(\mathbb{R}^d) : Lf = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left[a_{ij}^L \frac{\partial}{\partial x_j} f\right] + \sum_{i=1}^d b_i^L \frac{\partial}{\partial x_i} f,$$

The main result in [KS06a] states that the corresponding semigroup of transition kernels is intrinsically ultracontractive on D if $(a^L_{ij})_{i,j=1,\ldots,d}$ is symmetric as well as uniformly elliptic, and $a^L_{ij}, b^L_i \in C^\infty_b(\mathbb{R}^d)$, where ${}^{\partial b^L_i}/_{\partial x_i}$ is bounded, for all $i, j \in \{1, \ldots, d\}$. Note that our result cannot be compared directly with the result by by Kim and Song, because they do not prove the existence of a positive continous $\lambda_d|_U$–density under their assumptions. Many of our conditions on the coefficients as well as on the domain are necessary in order to obtain the function $p_{(\cdot)}(\cdot, \cdot)$ in Theorem 3.1, which is crucial for our considerations. Furthermore, Kim and Song consider semigroups of transition operators on $\mathscr{L}^1(D, \lambda_d|_D)$, whereas we consider semigroups of transition operators on $C_0(U)$, i.e., they consider a different situation than we do. In particular, this enables them to apply Jentzsch's theorem in order to obtain a positive normalised eigenfunction corresponding to the principal eigenvalue, whereas we refer to Theorem 5.5 and Theorem 6.1 in Chapter 3 in [Pin95] for this purpose. That we use the results by Pinsky is one reason why we cannot consider more general domains. However, our method has the virtue that the diffusion and drift coefficients do not have to consist of $C^\infty(\bar{U})$–functions, i.e., in this respect our main result is stronger than the main result in [KS06a]. Here we would like to point out that when comparing the assumptions on the coefficients one has to bear in mind that our operator T is given in nondivergence form, whereas Kim and Song consider an operator whose principal part is in divergence form. Finally, the method used by Kim and Song in [KS06a] is totally different from our approach, and thus our work adds a new approach to tackle the question of intrinsic ultracontractivity.

At the end of this chapter we would like to discuss briefly the assumptions on the domain and on the diffusion and drift coefficients of T. As we have seen in the previous chapter, we don't need such strong assumptions in order to obtain a diffusion. However, we need the conditions on the coefficients in order that we can use the results from [Dyn65II], [Pin95], [GQZ88], [HS82] as well as [Aro67]. The respective results are indispensable for our considerations, and hence we cannot relax the conditions on the coefficients. Note that the restriction that U is a bounded $C^{2,1}$–domain is necessary in order that we can apply Theorem 0.6 in the appendix of [Dyn65II], which requires a bounded $C^{1,\theta}$–domain for some $\theta > 0$, and that we can utilise the theorem in [HS82], which presumes a $C^{1,1}$–domain, as well as Theorem 5.5 and Theorem 6.1 in Chapter 3 in [Pin95], which require a $C^{2,\theta}$–domain for some $\theta > 0$.

Chapter 4

Uniform Conditional Ergodicity

The purpose of this chapter is to examine the significance of intrinsic ultracontractivity in stochastics. We will start with a brief outline of the theoretical background in order to point out the main underlying issues.

For the time being, consider a Markov semigroup $(P'_t)_{t\in\mathbb{R}_0^+}$ of kernels on some measurable space $(E,\mathscr{E})$. Then a probability measure ν on $(E,\mathscr{E})$ is called **stationary distribution** if

$$\forall\, t>0\,\forall\, B\in\mathscr{E}: \int_E \nu(\mathrm{d}x)P'_t(x,B)=\nu(B).$$

We say that $(P'_t)_{t\in\mathbb{R}_0^+}$ is **ergodic** if there exists a probability measure μ on $(E,\mathscr{E})$ with

$$\forall\, x\in E\,\forall\, B\in\mathscr{E}: P'_t(x,B)\to\mu(B) \text{ as } t\to\infty.$$

One can show that for any ergodic process the limit distribution μ is a stationary distribution. By applying the Dominated Convergence Theorem we obtain that an ergodic stochastic process has a unique stationary distribution, namely the limit distribution μ, since

$$\nu(B)=\lim_{t\to\infty}\int_E \nu(\mathrm{d}x)P'_t(x,B)\overset{\mathrm{DCT}}{=}\int_E \nu(\mathrm{d}x)\lim_{t\to\infty}P'_t(x,B)=\int_E \nu(\mathrm{d}x)\mu(B)=\mu(B)$$

for each $B\in\mathscr{E}$ and every stationary distribution ν on $\mathscr{E}$.

Now we will consider the conditional case. To this end let $(P'_t)_{t\in\mathbb{R}_0^+}$ be a sub–Markov semigroup of kernels on some measurable space $(E,\mathscr{E})$. Then we call a probability measure ν on $(E,\mathscr{E})$ **quasi–stationary distribution** if

$$\forall\, t>0\,\forall\, B\in\mathscr{E}: \frac{\int_E \nu(\mathrm{d}x)P'_t(x,B)}{\int_E \nu(\mathrm{d}x)P'_t(x,E)}=\nu(B),$$

Moreover, $(P'_t)_{t\in\mathbb{R}_0^+}$ is referred to as **conditionally ergodic** if there exists a probability measure μ on $(E,\mathscr{E})$ with

$$\forall\, x\in E\,\forall\, B\in\mathscr{E}: \frac{P'_t(x,B)}{P'_t(x,E)}\to\mu(B) \text{ as } t\to\infty. \tag{4.1}$$

As in the ergodic case one can show that the limit distribution μ for a conditionally ergodic semigroup is a quasi–stationary distribution. However, the main difference between the ergodic case and the conditionally ergodic case is that now the quasi–stationary distribution is not necessarily unique anymore. Indeed, S. Karlin, J. L. McGregor and E. van Doorn have given an example of a birth–death process on $\mathbb{N}_0 \cup \{-1\}$ with absorption at -1 and linear birth rates b_n and death rates d_n, given by

$$\forall\, n \in \mathbb{N}_0 : \quad b_n = (n+1)c_b \quad \text{and} \quad d_n = (n+1)c_d,$$

where $0 < c_b < c_d$, whose corresponding semigroup of transition kernels is conditionally ergodic but has a one–parametric family of quasi–stationary distributions. This example has been considered in [Sch03].

The possible non–uniqueness of a quasi–stationary distribution leads to the question under which assumptions does a unique quasi–stationary distribution exist. It will turn out that this is the case if the convergence in (4.1) is uniform in $x \in E$ and $B \in \mathscr{E}$, i.e., if $(P'_t)_{t\in\mathbb{R}_0^+}$ is **uniformly conditionally ergodic**. Thus, the question is: When is the convergence in (4.1) uniform in $x \in E$ and $B \in \mathscr{E}$? The aim of this chapter is to show how intrinsic ultracontractivity is related to this question.

Now we revert to the set–up of the previous chapter, which is a special case of the more general theoretical background related to the conditional case outlined above. Recall that $U \subseteq \mathbb{R}^d$ is a bounded $C^{2,1}$–domain, and that the diffusion semigroup $(P_t)_{t\in\mathbb{R}_0^+}$ is a sub–Markov semigroup on $(U, \mathscr{B}(U))$. This chapter is devoted to proving the following theorem, which states that intrinsic ultracontractivity implies uniqueness of the quasi–stationary distribution. We will prove this theorem by showing that intrinsic ultracontractivity implies that $(P_t)_{t\in\mathbb{R}_0^+}$ is uniformly conditionally ergodic.

Theorem 4.1 *If $(P_t)_{t\in\mathbb{R}_0^+}$ is intrinsically ultracontractive, then we have*

$$\exists\, t_0 > 0\, \exists\, \delta_{t_0}, \vartheta_{t_0} > 0\, \forall\, t \geq t_0\, \forall\, x \in U\, \forall\, B \in \mathscr{B}(U) : \left| \frac{P_t(x,B)}{P_t(x,U)} - \nu(B) \right| \leq \delta_{t_0} e^{-\vartheta_{t_0} t},$$

where $\nu \in \mathcal{P}(U)$ is defined by $\nu(B) = \int_B \psi \; d\lambda_d$ for all $B \in \mathscr{B}(U)$. Moreover, ν is the unique quasi–stationary distribution on $(U, \mathscr{B}(U))$.

Before we prove Theorem 4.1 we will provide some lemmas, which we will utilise in the proof.

Lemma 4.2 *If the diffusion is uniformly conditionally ergodic, i.e., if there exists a probability measure μ on $(U, \mathscr{B}(U))$ such that*

$$\frac{P_t(x,B)}{P_t(x,U)} \to \mu(B) \tag{4.2}$$

uniformly in $x \in U$ and $B \in \mathscr{B}(U)$ as $t \to \infty$, then the limit distribution μ is the unique quasi–stationary distribution on $(U, \mathscr{B}(U))$.

Proof Let $\varepsilon > 0$, and in view of the uniformity of the convergence in (4.2) choose some $t_\varepsilon > 0$ with

$$\forall\, x \in U : \left| \frac{P_{t_\varepsilon}(x,\cdot)}{P_{t_\varepsilon}(x,U)} - \mu \right| \le \varepsilon.$$

Let υ be some quasi–stationary distribution on $\mathscr{B}(U)$, then

$$\begin{aligned} |\upsilon(B) - \mu(B)| &= \frac{\left| \int_U \upsilon(\mathrm{d}x) \left(\frac{P_{t_\varepsilon}(x,B)}{P_{t_\varepsilon}(x,U)} - \mu(B) \right) P_{t_\varepsilon}(x,U) \right|}{\int_U \upsilon(\mathrm{d}x) P_{t_\varepsilon}(x,U)} \\ &\le \frac{\int_U \upsilon(\mathrm{d}x) \left| \frac{P_{t_\varepsilon}(x,B)}{P_{t_\varepsilon}(x,U)} - \mu(B) \right| P_{t_\varepsilon}(x,U)}{\int_U \upsilon(\mathrm{d}x) P_{t_\varepsilon}(x,U)} \\ &\le \varepsilon \end{aligned}$$

holds true for all $B \in \mathscr{B}(U)$. Since $\varepsilon > 0$ was chosen arbitrarily, this yields the assertion. □

Lemma 4.3 *If $(P_t)_{t\in\mathbb{R}_0^+}$ is intrinsically ultracontractive, then we have*

$$\forall\, s > 0\, \exists\, \rho_s > 0, \varrho_s > 0\, \forall\, t \ge s\, \forall\, x \in U : \rho_s \le \frac{e^{-\gamma t}\varphi(x)}{P_t(x,U)} \le \varrho_s.$$

Proof We have

$$\forall\, t > 0\, \forall\, x \in U : e^{-\gamma t}\varphi(x) = \int_U P_t(x,\mathrm{d}y)\varphi(y) \le \|\varphi\|_\infty P_t(x,U),$$

which gives an upper bound. By means of Theorem 3.8 (cf. the proof of "(i) $\Longrightarrow$ (ii)" in Theorem 3.10) the intrinsic ultracontractivity of $(P_t)_{t\in\mathbb{R}_0^+}$ yields

$$\forall\, s > 0\, \exists\, K_s \subseteq U, K_s \text{ compact}, \exists\, \kappa_{s,K_s} > 0\, \forall\, x \in U : P_s(x,K_s) \ge \kappa_{s,K_s} P_s(x,U).$$

Moreover, in conjunction with the Chapman–Kolmogorov Equation this results in

$$P_t(x,K_s) = \int_U P_{t-s}(x,\mathrm{d}y)P_s(y,K_s) \ge \kappa_{s,K_s} \int_U P_{t-s}(x,\mathrm{d}y)P_s(y,U) = \kappa_{s,K_s} P_t(x,U).$$

holds true for all $0 < s < t$ and any $x \in U$. Now, we obtain a lower bound, because

$$\begin{aligned} e^{-\gamma t}\varphi(x) = \int_U P_t(x,\mathrm{d}y)\varphi(y) &\ge \int_{K_s} P_t(x,\mathrm{d}y)\varphi(y) \\ &\ge \min_{y\in K_s} \varphi(y) P_t(x,K_s) \ge \min_{y\in K_s} \varphi(y)\kappa_{t,K_s} P_t(x,U). \end{aligned}$$

holds for all $0 < s \le t$ and any $x \in U$. □

Until the end of this chapter we fix some arbitrary $t_0 > 0$.

Lemma 4.4 *Let $B \in \mathscr{B}(U)$. If $(P_t)_{t\in\mathbb{R}_0^+}$ is intrinsically ultracontractive, then we have*

$$\forall\, s_0 > 0\,\exists\, \eta_{s_0} > 0\,\forall\, s \geq s_0\,\forall\, x, y \in U : \frac{\int_{U\setminus B} P_{t_0}(x, dz)P_s(z, U)}{P_{t_0+s}(x, U)} + \frac{\int_B P_{t_0}(y, dz)P_s(z, U)}{P_{t_0+s}(y, U)} \geq \eta_{s_0},$$

where η_{s_0} does not depend on B.

Proof Let $s_0 > 0$, then we infer from the previous lemma and Theorem 3.8 that

$$\begin{aligned}
&\frac{\int_{U\setminus B} P_{t_0}(x, \mathrm{d}z)P_s(z, U)}{P_{t_0+s}(x, U)} + \frac{\int_B P_{t_0}(y, \mathrm{d}z)P_s(z, U)}{P_{t_0+s}(y, U)}\\
\overset{T.3.8}{\geq}\;& \beta_{t_0} e^{\gamma t_0} \frac{e^{-\gamma(t_0+s)}\varphi(x)}{P_{t_0+s}(x, U)} e^{\gamma s} \int_{U\setminus B} P_s(z, U)\psi(z)\lambda_d(\mathrm{d}z)\\
&+\beta_{t_0} e^{\gamma t_0} \frac{e^{-\gamma(t_0+s)}\varphi(y)}{P_{t_0+s}(y, U)} e^{\gamma s} \int_B P_s(z, U)\psi(z)\lambda_d(\mathrm{d}z)\\
\overset{L.4.3}{\geq}\;& \beta_{t_0} e^{\gamma t_0} \rho_{t_0+s_0} \int_U e^{\gamma s} T_s^* \psi \,\mathrm{d}\lambda_d\\
\geq\;& \beta_{t_0} e^{\gamma t_0} \rho_{t_0+s_0} \int_U \psi \,\mathrm{d}\lambda_d =: \eta_{s_0}
\end{aligned}$$

holds for all $s \geq s_0$. □

Lemma 4.5 *If $(P_t)_{t\in\mathbb{R}_0^+}$ is intrinsically ultracontractive, then we have*

$$\exists\, \delta_{t_0}, \vartheta_{t_0} > 0\,\forall\, t \geq t_0 : \sup_{x,y\in U, B\in\mathscr{B}(U)} \left| \frac{P_t(x, B)}{P_t(x, U)} - \frac{P_t(y, B)}{P_t(y, U)} \right| \leq \delta_{t_0} e^{-\vartheta_{t_0} t}.$$

Proof Define a signed measure Q_s^{xy} on $\mathscr{B}(U)$ by

$$\forall\, B \in \mathscr{B}(U) : Q_s^{xy}(B) = \frac{\int_B P_{t_0}(x, \mathrm{d}z)P_s(z, U)}{P_{t_0+s}(x, U)} - \frac{\int_B P_{t_0}(y, \mathrm{d}z)P_s(z, U)}{P_{t_0+s}(y, U)}.$$

The Chapman–Kolmogorov Equation yields that

$$Q_s^{xy}(U) = \frac{P_{t_0+s}(x, U)}{P_{t_0+s}(x, U)} - \frac{P_{t_0+s}(y, U)}{P_{t_0+s}(y, U)} = 0. \tag{4.3}$$

Moreover, we infer from Lemma 4.4 that

$$Q_s^{xy}(B) = 1 - \frac{\int_{U\setminus B} P_{t_0}(x, \mathrm{d}z)P_s(z, U)}{P_{t_0+s}(x, U)} - \frac{\int_B P_{t_0}(y, \mathrm{d}z)P_s(z, U)}{P_{t_0+s}(y, U)} \leq 1 - \eta_{s_0}$$

holds for all $x, y \in U$, every $B \in \mathscr{B}(U)$, any $s_0 > 0$ and each $s \geq s_0$. We conclude that

$$\forall\, s_0 > 0\,\forall\, s \geq s_0 : \sup_{x,y\in U, B\in\mathscr{B}(U)} |Q_s^{xy}(B)| \leq 1 - \eta_{s_0} =: \xi_{s_0} < 1. \tag{4.4}$$

The following idea goes back to J. L. Doob. Let $B \in \mathscr{B}(U)$ and consider $f, g : \mathbb{R}^+ \to \mathbb{R}_0^+$, defined by

$$\forall\, t > 0 : \quad f(t) := \sup_{x \in U} \frac{P_t(x,B)}{P_t(x,U)} \quad \text{and} \quad g(t) := \inf_{x \in U} \frac{P_t(x,B)}{P_t(x,U)}.$$

Note that f is monotonically decreasing and g is monotonically increasing. Consequently, $h : \mathbb{R}^+ \to \mathbb{R}_0^+$, defined by

$$\forall\, t > 0 : h(t) = f(t) - g(t) = \sup_{x,y \in U} \left| \frac{P_t(x,B)}{P_t(x,U)} - \frac{P_t(y,B)}{P_t(y,U)} \right|,$$

is monotonically decreasing. Now consider the Hahn dethesis of U with respect to Q_s^{xy}, i.e., $U = U^+ \cup U^-$, where $Q_s^{xy}(U^+) \geq 0$, $Q_s^{xy}(U^-) \leq 0$ and $U^+ \cup U^- = \emptyset$. Note that $U^- = -U^+$. We infer from the Chapman–Kolmogorov Equation that

$$\begin{aligned}
& \frac{P_{t_0+s}(x,B)}{P_{t_0+s}(x,U)} - \frac{P_{t_0+s}(y,B)}{P_{t_0+s}(y,U)} \\
= \; & \int_U \frac{P_s(z,B)}{P_s(z,U)} \frac{P_s(z,U) P_{t_0}(x,\mathrm{d}z)}{P_{t_0+s}(x,U)} - \int_U \frac{P_s(z,B)}{P_s(z,U)} \frac{P_s(z,U) P_{t_0}(y,\mathrm{d}z)}{P_{t_0+s}(y,U)} \\
= \; & \int_U \frac{P_s(z,B)}{P_s(z,U)} Q_s^{xy}(\mathrm{d}z) \\
\leq \; & f(s) Q_s^{xy}(U^+) + g(s) Q_s^{xy}(U^-) \\
\overset{(*)}{=} \; & h(s) Q_s^{xy}(U^+) \\
\overset{(4.4)}{\leq} \; & h(s) \xi_{s_0}
\end{aligned}$$

holds true for all $x, y \in U$, every $s_0 > 0$ and any $s \geq s_0$. Therefore, $h(t_0 + s) \leq h(s)\xi_{s_0}$. Note that $(*)$ holds true, because, by (4.3), $Q_s^{xy}(U^+) + Q_s^{xy}(U^-) = Q_s^{xy}(U) = 0$. With $s_0 = t_0$ we infer

$$\forall\, n \in \mathbb{N} : h(nt_0) \leq \xi_{s_0}^{n-1} h(t_0) \leq \xi_{s_0}^{n-1}.$$

Now let $t \geq t_0$ and choose $n_t \in \mathbb{N}$ such that $n_t t_0 < t \leq (n_t + 1)t_0$, then $t/t_0 - 2 \leq n_t - 1$. Thus, since h is monotonically decreasing and $\xi_{s_0} < 1$, we deduce

$$h(t) \leq h(n_t t_0) \leq \xi_{t_0}^{n_t - 1} \leq \xi_{t_0}^{-2} \xi_{t_0}^{t/t_0}.$$

With $\delta_{t_0} := \xi_{t_0}^{-2}$ and $\vartheta_{t_0} := -(\ln(\xi_{t_0}^{1/t_0})$, this yields the assertion.

□

Lemma 4.6 *If $(P_t)_{t \in \mathbb{R}_0^+}$ is intrinsically ultracontractive, then the diffusion is conditionally ergodic with limit distribution $\mu \in \mathcal{P}(U)$. Moreover,*

$$\exists\, \delta_{t_0}, \vartheta_{t_0} > 0 \,\forall\, t \geq t_0 \,\forall\, x \in U : \left| \frac{P_t(x,\cdot)}{P_t(x,U)} - \mu \right| \leq \delta_{t_0} e^{-\vartheta_{t_0} t},$$

which in particular yields that the diffusion is in fact uniformly conditionally ergodic.

Proof We infer from the Chapman–Kolmogorov Equation and Lemma 4.5 that there exist $\delta_{t_0}, \vartheta_{t_0} > 0$ such that

$$\begin{aligned}
\left|\frac{P_{t+s}(x,B)}{P_{t+s}(x,U)} - \frac{P_t(x,B)}{P_t(x,U)}\right| &= \left|\frac{\int_U P_t(x,\mathrm{d}y)P_s(y,B) - \frac{P_t(x,B)}{P_t(x,U)}\int_U P_t(x,\mathrm{d}y)P_s(y,U)}{P_{s+t}(x,U)}\right| \\
&\leq \frac{\int_U \left|\frac{P_t(y,B)}{P_t(y,U)} - \frac{P_t(x,B)}{P_t(x,U)}\right| P_t(x,\mathrm{d}y)P_s(y,U)}{P_{t+s}(x,U)} \qquad (4.5)\\
&\overset{L.4.5}{\leq} \delta_{t_0}e^{-\vartheta_{t_0}t}
\end{aligned}$$

holds true for all $t \geq t_0$, $x, y \in U$ and $B \in \mathscr{B}(U)$, which shows that $P_t(x,B)P_t(x,U)^{-1}$ is Cauchy convergent as $t \to \infty$. Since $\mathbb{R}_0^+$ is complete, we deduce that there exists a measure $\mu : \mathscr{B}(U) \to \mathbb{R}_0^+$ with

$$\forall\, B \in \mathscr{B}(U) : \lim_{t\to\infty} \frac{P_t(x,B)}{P_t(x,U)} = \mu(B).$$

Now, our above considerations result in

$$\begin{aligned}
\forall\, s,t > 0\,\forall\, x \in U : \left|\frac{P_t(x,\cdot)}{P_t(x,U)} - \mu\right| &\leq \left|\frac{P_t(x,\cdot)}{P_t(x,U)} - \frac{P_{t+s}(x,\cdot)}{P_{t+s}(x,U)}\right| + \left|\frac{P_{t+s}(x,\cdot)}{P_{t+s}(x,U)} - \mu\right| \\
&\overset{(4.5)}{\leq} \delta_{t_0}e^{-\vartheta_{t_0}t} + \left|\frac{P_{t+s}(x,\cdot)}{P_{t+s}(x,U)} - \mu\right|,
\end{aligned}$$

which yields the assertion, because $|P_{t+s}(x,\cdot)P_{t+s}(x,U)^{-1} - \mu| \to 0$ as $s \to \infty$. □

Now we are in a position to reap the fruits of our work, because the previous lemmas enable us to prove Theorem 4.1. Because we have already done all the hard work in the preparations above, the proof of Theorem 4.1 is rather short.

Proof of Theorem 4.1 In view of Lemma 4.2 and Lemma 4.6 it only remains to show that ν is quasi–stationary. Recall that $\int_U p_t(\cdot,y)\psi\ \mathrm{d}\lambda_d = T_t^*\psi(y) = e^{-\gamma t}\psi(y)$ for all $y \in U$. Thus,

$$\int_U \nu(\mathrm{d}x)P_t(x,B) = \int_U\int_B p_t(x,y)\psi(x)\lambda_d(\mathrm{d}y)\lambda_d(\mathrm{d}x) = e^{-\gamma t}\int_B \psi\ \mathrm{d}\lambda_d = e^{-\gamma t}\nu(B)$$

holds true for every $B \in \mathscr{B}(U)$. Analogously we obtain by means of the normalisation of ψ (cf. Theorem 3.3) that $\int_U \nu(\mathrm{d}x)P_t(x,U) = e^{-\gamma t}\int_U \psi\ \mathrm{d}\lambda_d = e^{-\gamma t}$. It follows that

$$\forall\, B \in \mathscr{B}(U) : \frac{\int_U \nu(\mathrm{d}x)P_t(x,B)}{\int_U \nu(\mathrm{d}x)P_t(x,U)} = \nu(B),$$

which shows that ν is quasi–stationary. □

In conclusion we would like to point out that in Theorem 5 in [GQZ88] Gong, Qian and Zhao have shown that under our assumptions on U and T the semigroup $(Q_t)_{t\in\mathbb{R}_0^+}$, defined by

$$\forall\, t > 0\,\forall\, x,y \in U : Q_t(x,y) = e^{\gamma t}\frac{p_t(x,y)}{\varphi(x)\psi(y)},$$

is uniformly ergodic. This result does not depend on intrinsic ultracontractivity. In fact, their paper [GQZ88] does not deal with intrinsic ultracontractivity at all. That $(Q_t)_{t\in\mathbb{R}_0^+}$ is uniformly ergodic under the assumption of intrinsic ultracontractivity was shown by Kim and Song in Theorem 2.5 in [KS06a].

Appendix A

Definitions

In this appendix we will compile a few definitions which may be helpful for readers who are not familiar with the concepts utilised in this thesis. Throughout this appendix let $(E, \mathscr{B}(E))$ be a measurable space, where $(E, \mathcal{T})$ is a Polish space and $\mathscr{B}(E) := \sigma(\mathcal{T})$.

Definition A.1 *Let $X := (X_t)_{t\in\mathbb{R}_0^+}$ be a family of random variables on some measurable space $(\Omega, \mathscr{F})$ with values in $(E, \mathscr{B}(E))$. Let $(P^\nu)_{\nu\in\mathcal{P}(E)}$ be a family of probability measures on $(\Omega, \mathscr{F})$. For any $\nu \in \mathcal{P}(E)$ let $\sigma(X)_\nu$ denote the completion of $\sigma(X)$ with respect to P^ν. We denote by $\mathcal{N}_\nu$ the set of all P^ν–null sets in $\sigma(X)$, and for every $t \in \mathbb{R}_0^+$ we put $\mathscr{F}_t^\nu := \mathscr{F}_t^X \vee \mathcal{N}_\nu$. Now we obtain a complete filtration $\mathscr{G}^X := (\mathscr{G}_t^X)_{t\in\mathbb{R}_0^+}$, defined by*

$$\forall\, t \in \mathbb{R}_0^+ : \mathscr{G}_t^X := \bigcap_{\nu\in\mathcal{P}(E)} \mathscr{F}_t^\nu,$$

which we call the **completion** *of $\mathscr{F}^X$. Moreover, if X is right–continuous, then $\mathscr{G}^X$ can be shown to be right–continuous (cf. (2.10) Proposition in Chapter III in [RY99]), and in this case we call it the* **complete right–continuous filtration** *generated by X.*

Definition A.2 *Let $d \in \mathbb{N}$ and $\alpha := (\alpha_1, \ldots, \alpha_d)^T \in \mathbb{N}_0^n$ be a multi-index. For any $x \in \mathbb{R}^d$ we put*

$$|\alpha| := \sum_{i=1}^d \alpha_i, \quad \alpha! := \prod_{i=1}^d \alpha_i!, \quad x^\alpha := \prod_{i=1}^d x_i^{\alpha_i}.$$

Now for an arbitaray vector space X we define an d–dimensional **polynomial** *$\varphi : \mathbb{R}^d \to X$ with degree $k \in \mathbb{N}$ by*

$$\forall\, x \in \mathbb{R}^d : \varphi(x) = \sum_{\alpha:|\alpha|\le k} x^\alpha a_\alpha,$$

where $a_\alpha \in X$ for any multi-index α with $|\alpha| \le k$ and $a_\alpha \ne 0$ for some multi-index α with $|\alpha| = k$. If $X = \mathbb{R}$, then we will write $a_\alpha x^\alpha$ instead of $x^\alpha a_\alpha$. Furthermore, for any $f \in C^\infty(\mathbb{R}^d)$ we define the d–dimensional **Taylor series** *of f at $x_0 \in \mathbb{R}^n$ by*

$$\forall\, x \in \mathbb{R}^d : f(x) = \sum_\alpha \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha$$

A.1 Kernels

Definition A.3 *Let $(U, \mathscr{U})$ and $(V, \mathscr{V})$ be measurable spaces. A function $\mu : U \times \mathscr{V} \to \bar{\mathbb{R}}_0^+$ is called a* **(Markov) kernel** *from $(U, \mathscr{U})$ to $(V, \mathscr{V})$ if*

(i) $\mu(u, \cdot)$ is a (probability) measure on $\mathscr{V}$ for all $u \in U$,
(ii) $\mu(\cdot, B)$ is $\mathscr{U}$–$\mathscr{B}(\bar{\mathbb{R}}_0^+)$–measurable for all $B \in \mathscr{V}$.

A (Markov) kernel from $(U, \mathscr{U})$ to $(U, \mathscr{U})$ is said to be a (Markov) kernel on $(U, \mathscr{U})$. Furthermore, we call a kernel $\mu : U \times \mathscr{V} \to [0, 1]$ **sub–Markov kernel**.

Definition A.4 *Let $(U, \mathscr{U})$, $(V, \mathscr{V})$ and $(W, \mathscr{W})$ be measurable spaces. Furthermore, let μ be a kernel from $(U, \mathscr{U})$ to $(V, \mathscr{V})$ and ν a kernel from $(V, \mathscr{V})$ to $(W, \mathscr{W})$. Then we call the kernel $\mu \circ \nu$ from $(U, \mathscr{U})$ to $(W, \mathscr{W})$, defined by*

$$\forall\, u \in U\, \forall\, B \in \mathscr{W} : \mu \circ \nu(u, B) = \int_V \mu(u, dv)\nu(v, B),$$

the **composition** *of μ and ν.*

Definition A.5 *Let $(U, \mathscr{U})$, $(V, \mathscr{V})$ and $(W, \mathscr{W})$ be measurable spaces. Furthermore, let ν be a kernel from $(U, \mathscr{U})$ to $(V, \mathscr{V})$ and μ a kernel from $(V, \mathscr{V})$ to $(W, \mathscr{W})$. Then the kernel $\nu \otimes \mu$ from $(U, \mathscr{U})$ to $(V \times W, \mathscr{V} \otimes \mathscr{W})$, defined by*

$$\forall\, x \in U\, \forall\, A \in \mathscr{V}\, \forall\, B \in \mathscr{W} : \nu \otimes \mu(x, A \times B) = \int_A \nu(x, dy)\mu(y, B),$$

is called the **product** *of ν and μ.*

Definition A.6 *Let $(U, \mathscr{U})$ and $(V, \mathscr{V})$ be measurable spaces. Furthermore, let ν be a measure on $(U, \mathscr{U})$ and μ a kernel from $(U, \mathscr{U})$ to $(V, \mathscr{V})$. Then we call the measure $\nu \otimes \mu$ on $(U \times V, \mathscr{U} \otimes \mathscr{V})$, defined by*

$$\forall\, A \in \mathscr{U}\, \forall\, B \in \mathscr{V} : \nu \otimes \mu(A \times B) = \int_A \nu(dx)\mu(x, B),$$

the **product** *of ν and μ. Furthermore, we will use the following simplification:*

$$\forall\, B \in \mathscr{V} : \nu \otimes \mu(B) := \nu \otimes \mu(E \times B).$$

Definition A.7 *We say a family $(\mu_t)_{t \in \mathbb{R}_0^+}$ of kernels on $(E, \mathscr{B}(E))$ satisfies the* **Chapman–Kolmogorov equation** *if*

$$\forall\, s, t \in \mathbb{R}_0^+\, \forall\, B \in \mathscr{B}(E) : \mu_{s+t}(x, B) = \int_E \mu_s(x, dy)\mu_t(y, B)$$

A.2 Semigroup Theory

Definition A.8 *Let* $(S, *)$ *be any pair where* $* : S \times S \to S$ *is a binary operation on* S*, and* S *is a nonempty set containing an identity element,* id*, with respect to* $*$*, i.e.,* $A * \mathrm{id} = \mathrm{id} * A = A$ *for all* $A \in S$*. For any family* $(A_t)_{t\in\mathbb{R}_0^+} \subseteq S$ *the pair* $\left((A_t)_{t\in\mathbb{R}_0^+}, *\right)$ *is called a* **semigroup** *if* $A_0 = \mathrm{id}$ *and*

$$\forall\, s, t \in \mathbb{R}_0^+ : A_{s+t} = A_s * A_t.$$

Definition A.9 *A semigroup* $(T_t)_{t\in\mathbb{R}_0^+}$ *of* $\mathcal{B}(E)$*–valued positive contraction operators on some domain* $C_0(E) \subseteq \mathcal{D} \subseteq \mathcal{B}(E)$ *is called a* **Feller semigroup** *on* $\mathcal{D}$ *if*

(i) $\forall\, t \in \mathbb{R}_0^+ : T_t C_0(E) \subseteq C_0(E)$,

(ii) $\forall\, f \in \mathcal{D}\, \forall\, x \in E : T_t f(x) \to f(x)$ *as* $t \downarrow 0$.

Definition A.10 *Let* $(T_t)_{t\in\mathbb{R}_0^+}$ *be a Feller semigroup on* $C_0(E)$*. For any* $\alpha > 0$ *we call the* $C_0(E)$*–valued operator* R_α *on* $C_0(E)$*, defined by*

$$\forall\, f \in C_0(E)\, \forall\, x \in E : R_\alpha f(x) = \int_{\mathbb{R}_0^+} e^{-\alpha t} T_t f(x) \lambda(dt).$$

the **resolvent** *of* $(T_t)_{t\in\mathbb{R}_0^+}$.

In addition, we call the operator $T : \mathcal{D} \to C_0(E)$*, given by* $R_\alpha^{-1} = \alpha \cdot \mathrm{id} - T$ *on* $\mathcal{D}$ *for all* $\alpha > 0$*, the* **generator** *of* $(T_t)_{t\in\mathbb{R}_0^+}$*. Here* $\mathcal{D}$*, given by* $\mathcal{D} = R_\alpha C_0(E)$ *for some* $\alpha > 0$*, denotes the domain of* T*. Note that, by Theorem 17.4 in [Kal01] such an operator exists and* $\mathcal{D}$ *does not depend on* $\alpha > 0$.

Definition A.11 *Let* T *be a* $\mathcal{B}(E)$*–valued linear operator with domain* $\mathcal{D} \subseteq \mathcal{B}(E)$*. We say that* T *satisfies the* **positive maximum principle** *if*

$$\left[f \in \mathcal{D}, x \in E : f^+ \le f(x)\right] \Longrightarrow [Tf(x) \le 0].$$

Appendix B

Useful Results

In this appendix we will present a few results to which we refer throughout this thesis, but which are not directly relevant for our method. Many results will not be proven, but for each result we give a reference where the respective proof can be found in the literature.

B.1 Stopping Times

Let $(\Omega, \mathscr{F})$ be some measurable space and let $\mathscr{G} := (\mathscr{G}_t)_{t\in\mathbb{R}_0^+}$ be a filtration on $(\Omega, \mathscr{F})$. In this section we won't go into detail. The motivation for including this section is merely to clarify the notion of stopping times.

Definition B.1 *A $\mathscr{G}$–$\mathscr{B}(\bar{\mathbb{R}}_0^+)$–measurable random variable $\tau : \Omega \to \bar{\mathbb{R}}_0^+$ is referred to as $\mathscr{G}$–**stopping time** if*

$$\forall\, t \in \mathbb{R}_0^+ : \{\tau \leq t\} \in \mathscr{G}_t.$$

Definition B.2 *For a $\mathscr{G}$–stopping time τ,*

$$\mathscr{G}_\tau := \{A \in \mathscr{F} : A \cap \{\tau \leq t\} \in \mathscr{G}_t \,\forall\, t \in \mathbb{R}_0^+\}$$

*is said to be the σ–**algebra of the** τ–**past**.*

Lemma B.3 *For any $\mathscr{G}$–stopping time τ we have that*

(i) $\mathscr{G}_\tau$ is a σ–algebra.
(ii) τ is $\mathscr{G}_\tau$–$\mathscr{B}(\bar{\mathbb{R}}_0^+)$–measurable.
(iii) $\mathscr{G}_\tau = \mathscr{G}_t$ for $t \in \mathbb{R}_0^+$ with $\tau \equiv t$.

Proof Let $t \in \mathbb{R}_0^+$.

(i) $\Omega \in \mathscr{G}_\tau$, since $\Omega \cap \{\tau \leq t\} = \{\tau \leq t\} \in \mathscr{G}_t$.
Let $A \in \mathscr{G}_\tau$. Then $A^{\complement} \cap \{\tau \leq t\} = (A \cup \{\tau \leq t\}^{\complement})^{\complement} = ((A \cap \{\tau \leq t\}) \cup \{\tau \leq t\}^{\complement})^{\complement} \in \mathscr{G}_t$.
Let $A_1, A_2, \ldots \in \mathscr{G}_\tau$. Then $\{\tau \leq t\} \cap \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}}(A_n \cap \{\tau \leq t\}) \in \mathscr{G}_t$.

(ii) We have to show that $\{\tau \leq \alpha\} \cap \{\tau \leq t\} \in \mathscr{G}_t$ for all $\alpha \in \mathbb{R}, t \in \mathbb{R}_0^+$. Firstly assume that $t \leq \alpha$. Then

$$\{\tau \leq \alpha\} \cap \{\tau \leq t\} = \{\tau \leq t\} \in \mathscr{G}_t,$$

since τ is a $\mathscr{G}$–stopping time. Conversely presume that $\alpha < t$. Then

$$\{\tau \leq \alpha\} \cap \{\tau \leq t\} = \{\tau \leq \alpha\} \in \mathscr{G}_\alpha \subseteq \mathscr{G}_t.$$

(iii) It is easy to see that

$$\mathscr{G}_t \subseteq \mathscr{G}_\tau,$$

and thus we only have to show that $\mathscr{G}_\tau \subseteq \mathscr{G}_t$. Let $t \in \mathbb{R}_0^+$, $\tau \equiv t$ and $A \in \mathscr{G}_\tau$. Then

$$A = A \cap \{\tau \leq t\} \in \mathscr{G}_t.$$

□

B.2 Markov Processes

Markov processes play a crucial role troughout this thesis. Firstly because diffusion processes are examples of Markov processes, and secondly because we deal with more general Markov processes in Chapter 1 in order to develop Feller diffusions. However, many results, which are helpful but not directly related to our proceeding in Chapter 1, will be presented here, in order to focus on the main ideas in Chapter 1.

Throughout this section let $(E, \mathscr{B}(E))$ be a measurable space, where $(E, \mathcal{T})$ is a Polish space and $\mathscr{B}(E) := \sigma(\mathcal{T})$.

B.2.1 Construction of Canonical Markov Processes

In Chapter 1 we provide the main concepts relating to the theory of Markov processes. Our approach in Chapter 1 is to start with a Markov semigroup and to construct a corresponding Markov process. The construction of canonical Markov processes, associated with Markov semigroups, via Kolmogorov's extension theorem is standard in the literature, but since our considerations rest upon this construction, we will provide a brief presentation of that approach.

Definition B.4 *Let $\mathscr{G} := (\mathscr{G}_t)_{t\in\mathbb{R}_0^+}$ be a filtration on some probability space $(\Omega, \mathscr{F}, P)$, and let $X := (X_t)_{t\in\mathbb{R}_0^+}$ be a $\mathscr{G}$–adapted E–valued stochastic process. We say that $(X, \mathscr{G})$ is a*

(i) **Markov process** *if*

$$\forall\, B \in \mathscr{B}(E)\,\forall\, s,t \in \mathbb{R}_0^+, s \leq t : P(X_t \in B|\mathscr{G}_s) = P(X_t \in B|X_s) \quad P\text{–a.s.}. \tag{B.1}$$

(ii) **strong Markov process** *if*

$$\forall\, B \in \mathscr{B}(E)\,\forall\, \tau \in \mathcal{S}_f(P,\mathscr{G})\forall\, t \in \mathbb{R}_0^+ : P(X_{\tau+t} \in B|\mathscr{G}_\tau) = P(X_{\tau+t} \in B|X_\tau) \quad P\text{–a.s.}. \tag{B.2}$$

If (B.1) [resp. (B.2)] holds and if $\mathscr{G}$ is the natural filtration with respect to X, then we may simply say X is a [strong] Markov process and omit mentioning the filtration.

Let ν be a measure on $(E, \mathscr{B}(E))$, and let $(\mu_t)_{t\in\mathbb{R}_0^+}$ be a semigroup of kernels on $(E, \mathscr{B}(E))$. For every $J = (t_1, \ldots, t_n) \in \mathcal{H}(\mathbb{R}_0^+)$ we define a measure μ_J on $(E^{n+1}, \mathscr{B}(E^{n+1}))$ by

$$\mu_J := \nu \otimes \mu_{t_1} \otimes \mu_{t_2-t_1} \otimes \ldots \otimes \mu_{t_n-t_{n-1}}. \tag{B.3}$$

Then we have that

$$\mu_J(B) = \underbrace{\int_E \ldots \int_E}_{n+1 \text{ times}} \nu(\mathrm{d}x_0)\mu_{t_1}(x_0, \mathrm{d}x_1)\ldots\mu_{t_n-t_{n-1}}(x_{n-1}, \mathrm{d}x_n)\mathbb{1}_B(x_0, \ldots, x_n)$$

holds for all $B \in \mathscr{B}(E^{n+1})$.

Definition B.5 *For any $J, K \in \mathcal{H}(\mathbb{R}_0^+)$, $J \subseteq K$, we define the* **projection** $\pi_{J,K} : E^K \to E^J$ *by*

$$\forall\, (x_i)_{i\in K} \in E^K : \pi_{J,K}((x_i)_{i\in K}) = (x_i)_{i\in J},$$

and for every $J \in \mathcal{H}(\mathbb{R}_0^+)$ we obtain the **projection** $\pi_J : E^{\mathbb{R}_0^+} \to E^J$ *by*

$$\forall\, (x_t)_{t\in\mathbb{R}_0^+} \in E^{\mathbb{R}_0^+} : \pi_J((x_t)_{t\in\mathbb{R}_0^+}) = (x_i)_{i\in J}$$

In case that $J = \{t\}$ for some $t \in \mathbb{R}_0^+$ we will also write π_t instead of $\pi_{\{t\}}$.

Definition B.6 *A family $(\mu_J)_{J\in\mathcal{H}(\mathbb{R}_0^+)}$ of probability measures μ_J on $(E^J, \mathscr{B}(E^J))$ is called* **projective** *if*

$$\forall\, J, K \in \mathcal{H}(\mathbb{R}_0^+), J \subsetneq K : \mu_K \circ \pi_{J,K}^{-1} = \mu_J.$$

Theorem B.7 *Let ν be a probability measure on some probability space $(\Omega, \mathscr{F})$ and let $(P_t)_{t\in\mathbb{R}_0^+}$ be a Markov semigroup on $(E, \mathscr{B}(E))$. The family $(\mu_J)_{J\in\mathcal{H}(\mathbb{R}_0^+)}$ of probability measures on $(E^{n+1}, \mathscr{B}(E^{n+1}))$ defined by (B.3) is projective.*

Proof Let $J := \{t_1 < \ldots < t_m\}, K := (t_1, \ldots, t_j, t, t_{j+1}, \ldots, t_m) \in \mathcal{H}(\mathbb{R}_0^+)$. We show that $\mu_K \circ \pi_{J,K}^{-1} = \mu_J$, which proves the assertion. Let $B_0, \ldots, B_m \in \mathscr{B}(E)$, then

$$\begin{aligned}
& \mu_K \circ \pi_{J,K}^{-1}(B_0 \times \ldots \times B_m) \\
= \;& \mu_K(B_0 \times \ldots \times B_j \times E \times B_{j+1} \ldots \times B_m) \\
= \;& \int_{B_m} \ldots \int_E \ldots \int_{B_0} \\
& \nu(\mathrm{d}x_0)P_{t_1}(x_0, \mathrm{d}x_1)\ldots P_{t-t_j}(x_j, \mathrm{d}x)P_{t_{j+1}-t}(x, \mathrm{d}x_{j+1})\ldots P_{t_m-t_{m-1}}(x_{m-1}, \mathrm{d}x_m) \\
\overset{(*)}{=} \;& \int_{B_m} \ldots \int_{B_{j+1}} \ldots \int_{B_0} \nu(\mathrm{d}x_0)P_{t_1}(x_0, \mathrm{d}x_1)\ldots P_{t_{j+1}-t_j}(x_j, \mathrm{d}x_{j+1})\ldots P_{t_m-t_{m-1}}(x_{m-1}, \mathrm{d}x_m) \\
= \;& \nu \otimes P_{t_1} \otimes P_{t_2-t_1} \otimes \ldots \otimes P_{t_m-t_{m-1}}(B_0 \times \ldots \times B_m)
\end{aligned}$$

$$= \mu_J(B_0 \times \ldots \times B_m),$$

where $(*)$ holds, since we have by the Chapman-Kolmogorov equation that

$$\int_E P_{t-t_j}(x_j, \mathrm{d}x) P_{t_{j+1}-t}(x, \mathrm{d}x_{j+1}) = P_{t_{j+1}-t_j}(x_j, \mathrm{d}x_{j+1}).$$

Now we obtain that $\mu_K \circ \pi_{J,K}^{-1} = \mu_J$. □

Theorem B.8 (Kolmogorov's Extension Theorem) *Let $(E, \mathscr{B}(E))$ be a Polish space. For any projective family $(\mu_J)_{J \in \mathcal{H}(\mathbb{R}_0^+)}$ of probability measures on $(E^J, \mathscr{B}(E^J))$ there exists a uniquely defined probability measure P on $(E^{\mathbb{R}_0^+}, \mathscr{Z}(E^{\mathbb{R}_0^+}))$ such that $P \circ \pi_J^{-1} = \mu_J$ for all $J \in \mathcal{H}(\mathbb{R}_0^+)$.*

Proof See 1.1.10 Theorem in [SV79]. □

Theorem B.9 *Let $(P_t)_{t \in \mathbb{R}_0^+}$ be a Markov semigroup on $(E, \mathscr{B}(E))$, let $\nu \in \mathcal{P}(E)$ and let $\pi_t : E^{\mathbb{R}_0^+} \to E$, $t \in \mathbb{R}_0^+$, be the projection as in Definition B.5. Then there exists a uniquely defined probability measure P^ν on $(\Omega, \mathscr{F}) := (E^{\mathbb{R}_0^+}, \mathscr{Z}(E^{\mathbb{R}_0^+}))$ such that the finite dimensional distributions of the coordinate mapping process $(X_t)_{t \in \mathbb{R}_0^+}$ on $(\Omega, \mathscr{F}, P^\nu)$, defined by $X_t := \pi_t$, are given by*

$$P^\nu_{(X_{t_1}, \ldots, X_{t_n})} = \nu \otimes P_{t_1} \otimes P_{t_2 - t_1} \otimes \ldots \otimes P_{t_n - t_{n-1}}$$

for each $(t_1, \ldots, t_n) \in \mathcal{H}(\mathbb{R}_0^+)$.

Proof For any $J = (t_1, \ldots, t_n) \in \mathcal{H}(\mathbb{R}_0^+)$ consider $\mu_J := \nu \otimes P_{t_1} \otimes P_{t_2 - t_1} \otimes \ldots \otimes P_{t_n - t_{n-1}}$. Then Theorem B.7 yields that $(\mu_J)_{J \in \mathcal{H}(\mathbb{R}_0^+)}$ is a projective family of probability measures, and the assertion follows from Kolmogorov's extension theorem (cf. Theorem B.8). □

The following theorem shows that the stochastic process which we have considered in the previous theorem turns out to be a Markov process. We give a proof which is more circuitous than necessary in order to prove the theorem itself, for that we prove an assertion for all $n \in \mathbb{N}$ which in fact, for the time being, we only need to show for $n = 1$. The reason is that later on we will need the result for all $n \in \mathbb{N}$, and thus we do it in more detail now and we will refer to this proof again at a later point of time.

Theorem B.10 *Let $(P_t)_{t \in \mathbb{R}_0^+}$ be a Markov semigroup on $(E, \mathscr{B}(E))$, let $\nu \in \mathcal{P}(E)$ and let $(\Omega, \mathscr{F}, P^\nu)$ as well as $X := (X_t)_{t \in \mathbb{R}_0^+}$ be as in Theorem B.9. Furthermore, let $(\mathscr{F}_t)_{t \in \mathbb{R}_0^+}$ be the natural filtration with respect to X. Then X is a Markov process with respect to P^ν.*

Proof For any $J = (t_1, \ldots, t_n) \in \mathcal{H}(\mathbb{R}_0^+)$ choose some $t_0 \le t_1$. Firstly, we want to show that

$$P^\nu((\pi_J \circ X)^{-1}(B_1, \ldots, B_n) | \mathscr{F}_{t_0}) = P_{t_1 - t_0} \otimes \ldots \otimes P_{t_n - t_{n-1}}(X_{t_0}, B_1 \times \ldots \times B_n) \quad P^\nu\text{–a.s.}$$

holds for all $B_1, \ldots, B_n \in \mathscr{B}(E)$. To this end observe that $P_{t_1 - s} \otimes \ldots \otimes P_{t_n - t_{n-1}}(X_{t_0}, B_1 \times \ldots \times B_n)$ is $\mathscr{F}_{t_0}$–$\mathscr{B}([0,1])$–measurable, and note that for every $s \in \mathbb{R}_0^+$ the σ–algebra $\mathscr{F}_{t_0}$ is generated by

$$\mathcal{G}_{\le t_0} := \Big\{ \{X_{t_{-m}} \in B_{-m}, \ldots, X_{t_{-1}} \in B_{-1}\} :$$

$$m \in \mathbb{N}, 0 < t_{-m} < \ldots < t_{-1} \le t_0, B_j \in \mathscr{B}(E), j = t_{-m}, \ldots, t_{-1}\Big\},$$

which contains Ω and is closed under the formation of finite intersections. Thus, we have to show that

$$\int_G \mathbb{1}_{(\pi_J \circ X)^{-1}(B_1,\ldots,B_n)}\, \mathrm{d}P^\nu = \int_G P_{t_1-t_0} \otimes \ldots \otimes P_{t_n-t_{n-1}}(X_{t_0}, B_1 \times \ldots \times B_n)\, \mathrm{d}P^\nu \tag{B.4}$$

holds for all $G \in \mathcal{G}_{\le t_0}$. Let $G = \{X_{t_{-m}} \in B_{-m}, \ldots, X_{t_{-1}} \in B_{-1}, X_{t_0} \in B_0\} \in \mathcal{G}_{\le t_0}$, then

$$\begin{aligned}
&\int_G \mathbb{1}_{(\pi_J \circ X)^{-1}(B_1,\ldots,B_n)}\, \mathrm{d}P^\nu \\
=\;& P^\nu_{(X_{t_{-m}},\ldots,X_{t_0},\ldots,X_{t_n})}(B_{-m} \times \ldots \times B_0 \times \ldots \times B_n) \\
=\;& \int_{B_n} \ldots \int_{B_{-m+1}} \int_{B_{-m}} \nu(\mathrm{d}x_{-m}) P_{t_{-m+1}-t_{-m}}(x_{-m}, \mathrm{d}x_{-m+1}) \ldots P_{t_n-t_{n-1}}(x_{n-1}, \mathrm{d}x_n) \\
=\;& \int_{B_0} \ldots \int_{B_{-m+1}} \int_{B_{-m}} \nu(\mathrm{d}x_{-m}) P_{t_{-m+1}-t_{-m}}(x_{-m}, \mathrm{d}x_{-m+1}) \\
& \ldots P_{t_0-t_{-1}}(x_{-1}, \mathrm{d}x_0) P_{t_1-t_0} \otimes \ldots \otimes P_{t_n-t_{n-1}}(x_0, B_1 \times \ldots \times B_n) \\
=\;& \int_G P_{t_1-t_0} \otimes \ldots \otimes P_{t_n-t_{n-1}}(X_{t_0}, B_1 \times \ldots \times B_n)\, \mathrm{d}P^\nu.
\end{aligned}$$

Therefore, (B.4) holds true. Moreover, since $\sigma(X_{t_0}) \subseteq \mathscr{F}_{t_0}$, we infer from $P_{t_1-t_0} \otimes \ldots \otimes P_{t_n-t_{n-1}}(X_{t_0}, B_1 \times \ldots \times B_n)$ being $\sigma(X_{t_0})$–$\mathscr{B}([0,1])$–measurable that

$$\begin{aligned}
P^\nu((\pi_J \circ X)^{-1}(B_1,\ldots,B_n)|\mathscr{F}_{t_0}) &= P_{t_1-t_0} \otimes \ldots \otimes P_{t_n-t_{n-1}}(X_{t_0}, B_1 \times \ldots \times B_n) \quad P^\nu\text{–a.s.} \\
&= P^\nu((\pi_J \circ X)^{-1}(B_1,\ldots,B_n)|X_{t_0}) \quad P^\nu\text{–a.s..}
\end{aligned} \tag{B.5}$$

For $n = 1$ this yields (B.1). □

Utilising the denotations as above, we call (X, P^ν) the **canonical Markov process** with respect to $(P_t)_{t\in\mathbb{R}_0^+}$ and with initial distribution $\nu \in \mathcal{P}(E)$ (cf. Definition 1.6).

B.2.2 Results relating to Canonical Markov Processes

For the remainder of this section let $(P_t)_{t\in\mathbb{R}_0^+}$ be a Markov semigroup on $(E, \mathscr{B}(E))$, and for any $\nu \in \mathcal{P}(E)$ let (X, P^ν) be the canonical Markov process on $(\Omega, \mathscr{F}) := (E^{\mathbb{R}_0^+}, \mathscr{Z}(E^{\mathbb{R}_0^+}))$ with respect to $(P_t)_{t\in\mathbb{R}_0^+}$ and with initial distribution ν. In addition, let $\mathscr{F}^X_{\ge t}$ and the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_0^+}$ be as in Definition 1.2. The aim of this subsection is to derive a few properties of canonical Markov processes which are utilised throughout this thesis.

Remark B.11 For all $x \in E$, $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}_0^+$ and $A, B_1, \ldots, B_n \in \mathscr{B}(E)$ we have

$$P^x_{X_{t_1},\ldots,X_{t_n}}(A \times B_1 \times \ldots \times B_n) = \delta_x \otimes P_{t_1} \otimes \ldots \otimes P_{t_n}(A \times B_1 \times \ldots \times B_n)$$

$$
\begin{aligned}
&= \int_A \delta_x(\mathrm{d}z) P_{t_1} \otimes \ldots \otimes P_{t_n}(z, B_1 \times \ldots \times B_n) \\
&= P_{t_1} \otimes \ldots \otimes P_{t_n}(x, B_1 \times \ldots \times B_n) \mathbb{1}_A(x).
\end{aligned}
$$

$\Diamond$

Lemma B.12 *Let P and Q be finite measures on some measurable space $(\Omega, \mathscr{F})$. Furthermore, let $\mathcal{A} \subseteq \mathscr{F}$ be closed under the formation of finite intersections, and presume that $P = Q$ on $\mathcal{A} \cup \{\Omega\}$. Then $P = Q$ on $\sigma(\mathcal{A})$.*

Proof See Theorem 3.3 in [Bil95]. There the assertion is shown for probability measures. However, the proof of Theorem 3.3 in [Bil95] can be extended to the assertion of Lemma B.12. $\square$

For any $t \in \mathbb{R}_0^+$ we define

$$
\mathcal{G}_{\geq t} := \Big\{ \{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\} : n \in \mathbb{N}, t \leq t_1 < \ldots < t_n, B_j \in \mathscr{B}(E), j = 1, \ldots, n \Big\}.
$$

Lemma B.13 *For fixed $\nu \in \mathcal{P}(E)$, the property (B.1), with P^ν substituted for P, is equivalent to any of the following statements:*

$$
\forall\, t \in \mathbb{R}_0^+ \,\forall\, A \in \mathscr{F}_{\geq t}^X : P^\nu(A|\mathscr{F}_t) = P^\nu(A|X_t) \quad P^\nu\text{–a.s.,} \tag{B.6}
$$

$$
\begin{aligned}
&\forall\, n \in \mathbb{N} \,\forall\, B_1, \ldots, B_n \in \mathscr{B}(E) \,\forall\, s_1, \ldots, s_n, t \in \mathbb{R}_0^+ : \\
&P^\nu(\theta_t^{-1}(X_{s_1}^{-1}(B_1), \ldots, X_{s_n}^{-1}(B_n))|\mathscr{F}_t) = P^{X_t}(X_{s_1}^{-1}(B_1), \ldots, X_{s_n}^{-1}(B_n)) \quad P^\nu\text{–a.s.,}
\end{aligned} \tag{B.7}
$$

$$
\forall\, A \in \sigma(X) \,\forall\, t \in \mathbb{R}_0^+ : P^\nu(\theta_t^{-1}(A)|\mathscr{F}_t) = P^{X_t}(A) \quad P^\nu\text{–a.s.,} \tag{B.8}
$$

$$
\forall\, Y \in \mathcal{B}(\Omega, \sigma(X)) \,\forall\, t \in \mathbb{R}_0^+ : E_\nu(Y \circ \theta_t|\mathscr{F}_t) = E_{X_t}(Y) \quad P^\nu\text{–a.s..} \tag{B.9}
$$

Proof "(B.1) $\Longrightarrow$ (B.6):" Let $t \in \mathbb{R}_0^+$. According to (B.5) we have $P^\nu(G|\mathscr{F}_t) = P^\nu(G|X_t)$ P^ν–a.s. for all $G \in \mathcal{G}_{\geq t}$. Hence the assertion follows from Lemma B.12.

"(B.6) $\Longrightarrow$ (B.7):" Let $n \in \mathbb{N}$, $B_1, \ldots, B_n \in \mathscr{B}(E)$ and $s_1, \ldots, s_n, t \in \mathbb{R}_0^+$. Firstly, note that

$$
\begin{aligned}
P^{X_t}((X_{s_1}, \ldots, X_{s_n})^{-1}(B_1 \times \ldots \times B_n)) &= P_{s_1} \otimes \ldots \otimes P_{s_n}(X_t, B_1 \times \ldots \times B_n) \\
&\overset{(B.5)}{=} P^\nu((X_{s_1+t}, \ldots X_{s_n+t})^{-1}(B_1 \times \ldots \times B_n)|X_t)
\end{aligned}
$$

holds true P^ν–a.s.. Furthermore,

$$
\begin{aligned}
\theta_t^{-1} \circ (X_{s_1}, \ldots, X_{s_n})^{-1}(B_1 \times \ldots \times B_n) &= ((X_{s_1}, \ldots, X_{s_n}) \circ \theta_t)^{-1}(B_1 \times \ldots \times B_n) \\
&= (X_{s_1+t}, \ldots, X_{s_n+t})^{-1}(B_1 \times \ldots \times B_n) \\
&\in \mathscr{F}_{\geq t}^X,
\end{aligned}
$$

and thus (B.6) yields the assertion.

"(B.7) $\Longrightarrow$ (B.8):" Let $A \in \sigma(X)$ and $t \in \mathbb{R}_0^+$. Analoguously to above we infer from (B.7) that $P^{X_t}(G) = P^\nu(\theta_t^{-1}(G)|\mathscr{F}_t)$ P^ν–a.s. for all $G \in \mathcal{G}_{\geq 0}$, and we obtain the assertion by means of Lemma B.12.

"(B.8) $\Longrightarrow$ (B.9):" Let $t \in \mathbb{R}_0^+$. Note that the left hand side in (B.9) is defined, because $Y \circ \theta_t \in \mathscr{L}^1(\Omega)$ for any $Y \in \mathcal{B}(\Omega, \sigma(X))$. The assertion can be proven by algebraic induction. By means of (B.8) the assertion is established if $Y = \mathbb{1}_A$ for some $A \in \sigma(X)$, and thus the assertion holds true if Y is a nonnegative simple function on $(\Omega, \sigma(X))$. Now assume that $Y \in \mathcal{B}(\Omega, \sigma(X))$ is nonnegative. Then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of nonnegative simple functions on $(\Omega, \sigma(X))$ such that $\varphi_n \uparrow Y$ as $n \to \infty$. Consequently,

$$E_\nu(Y \circ \theta_t|\mathscr{F}_t) = \lim_{n\to\infty} E_\nu(\varphi_n \circ \theta_t|\mathscr{F}_t) = \lim_{n\to\infty} E_{X_t}(\varphi_n) = E_{X_t}(Y) \quad P^\nu\text{–a.s..}$$

For arbitrary $Y \in \mathcal{B}(\Omega, \sigma(X))$ consider the positive part $Y^+ := Y \vee \mathbf{0}$ and the negative part $Y^- := (-Y) \vee \mathbf{0}$. Then we have

$$E_\nu(Y \circ \theta_t|\mathscr{F}_t) = E_\nu(Y^+ \circ \theta_t|\mathscr{F}_t) - E_\nu(Y^- \circ \theta_t|\mathscr{F}_t) = E_{X_t}(Y^+) - E_{X_t}(Y^-) = E_{X_t}(Y) \quad P^\nu\text{–a.s.,}$$

which proves the assertion.

"(B.9) $\Longrightarrow$ (B.1):" Let $s, t \in \mathbb{R}_0^+$, $B \in \mathscr{B}(E)$ and $Y := \mathbb{1}_{X_s^{-1}(B)} \in \mathcal{B}(\Omega, \sigma(X))$. Then

$$\begin{aligned} E_\nu(Y \circ \theta_t|\mathscr{F}_t) = E_{X_t}(Y) \quad P^\nu\text{–a.s.} &\Longrightarrow E_\nu(\mathbb{1}_{\theta_t^{-1} \circ X_s^{-1}(B)}|\mathscr{F}_t) = E_{X_t}(\mathbb{1}_{X_s^{-1}(B)}) \quad P^\nu\text{–a.s.} \\ &\Longrightarrow P^\nu(X_{s+t} \in B|\mathscr{F}_t) = P^{X_t}(X_s \in B) \quad P^\nu\text{–a.s.,} \end{aligned}$$

and $P^{X_t}(X_s \in B) = P_s(X_t, B) \overset{(B.5)}{=} P^\nu(X_{s+t} \in B|X_t)$ P^ν–a.s.. □

Corollary B.14 *In particular, Lemma B.13 shows that $(P^\nu)_{\nu\in\mathcal{P}(E)}$ satisfies the Markov property.*

Lemma B.15 *Let $\nu \in \mathcal{P}(E)$, $n \in \mathbb{N}$ and $f \in \mathcal{B}(E^n, \mathscr{B}(E^n))$. Then*

$$\int_A f \circ (X_{t_1+h}, \ldots, X_{t_n+h}) \; dP^\nu = \int_A E_{X_h}(f \circ (X_{t_1}, \ldots, X_{t_n})) \; dP^\nu.$$

holds for all $h, t_1, \ldots, t_n \in \mathbb{R}_0^+$ and for each $A \in \mathscr{F}_h$.

Proof Let $h, t_1, \ldots, t_n \in \mathbb{R}_0^+$, put $Y := f \circ (X_{t_1}, \ldots, X_{t_n})$ and note that $Y \in \mathcal{B}(\Omega, \sigma(X))$. Moreover, $f \circ (X_{t_1+h}, \ldots, X_{t_n+h}) = Y \circ \theta_h$, and thus we infer for any $A \in \mathscr{F}_h$ that

$$\begin{aligned} \int_A f \circ (X_{t_1+h}, \ldots, X_{t_n+h}) \, \mathrm{d}P^\nu &= \int_A E_\nu(Y \circ \theta_h|\mathscr{F}_h) \, \mathrm{d}P^\nu \overset{(B.9)}{=} \int_A E_{X_h}(Y) \, \mathrm{d}P^\nu \\ &= \int_A E_{X_h}(f \circ (X_{t_1}, \ldots, X_{t_n})) \, \mathrm{d}P^\nu. \end{aligned}$$

□

Corollary B.16 *Let $(T_t)_{t\in\mathbb{R}_0^+}$ be the semigroup of the transition operators T_t with respect to P_t, $t \in \mathbb{R}_0^+$, and let $f \in \mathcal{B}(E)$. Then*

$$\forall\, \nu \in \mathcal{P}(E)\, \forall\, t,h \in \mathbb{R}_0^+\, \forall\, A \in \mathscr{F}_h : \int_A T_t f \circ X_h \; dP^\nu = \int_A f \circ X_{t+h} \; dP^\nu.$$

Proof Let $t,h \in \mathbb{R}_0^+$. Then

$$T_t f \circ X_h = \int_E f(y) P_t(X_h, \mathrm{d}y) = \int_E f \; \mathrm{d}P^{X_h}_{X_t} = \int_\Omega f \circ X_t \; \mathrm{d}P^{X_h} = E_{X_h}(f \circ X_t),$$

and thus the assertion is a direct consequence of Lemma B.15. □

By means of Lemma 1.11 we have that Lemma B.15 und Corollary B.16 also hold if we replace the determinstic times by stopping times which are P^x–a.s. finite for any $x \in E$.

Lemma B.17 *We have that*

$$E_\nu(Y) = \int_E \nu(dx) E_x(Y).$$

holds for any $Y \in \mathcal{B}(\Omega, \sigma(X))$ and every $\nu \in \mathcal{P}(E)$.

Proof Let $Y \in \mathcal{B}(\Omega, \sigma(X))$ and $\nu \in \mathcal{P}(E)$. Since $P^\nu = \int_E \nu(\mathrm{d}x) P^x$, we deduce that

$$E_\nu(Y) = \int_\Omega Y \mathrm{d}P^\nu \overset{\text{DCT}}{=} \int_\Omega Y \int_E \nu(\mathrm{d}x) \mathrm{d}P^x = \int_E \nu(\mathrm{d}x) \int_\Omega Y \mathrm{d}P^x = \int_E \nu(\mathrm{d}x) E_x(Y).$$

□

B.3 Further Adjuvant Results

Theorem B.18 (Optional Sampling Theorem) *Let $\mathscr{F} := (\mathscr{F}_t)_{t\in\mathbb{R}_0^+}$ be a filtration on some probability space $(\Omega, \mathscr{G}, P)$, and let $X := (X_t)_{t\in\mathbb{R}_0^+}$ be a right–continuous martingale with respect to $\mathscr{F}$. Furthermore, let σ and τ be bounded $\mathscr{F}$–stopping times with $\sigma \le \tau$. Then*

$$E(X_\tau | \mathscr{F}_\sigma) = X_\sigma \quad P\text{–a.s.}.$$

The statement extends to unbounded stopping times iff $(X_t^+)_{t\in\mathbb{R}_0^+}$ is uniformly P–integrable.

Proof See Theorem 6.29 in [Kal01]. □

Theorem B.19 *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Furthermore, let $(E, \mathscr{B}(E))$ be some measurable space, where E is a Polish space. Let $X : \Omega \to E$ be some random variable. Then for every sub–σ–algebra $\mathscr{G} \subseteq \mathscr{F}$ there exists a regular $\mathscr{G}$–conditional distribution $P_{X|\mathscr{G}}$ of X. Moreover, if $P_{X|\mathscr{G}}$ and $P'_{X|\mathscr{G}}$ are regular $\mathscr{G}$–conditional distributions of X, then there exists some P–null set $N \in \mathscr{F}$ such that*

$$\forall\, \omega \in \Omega \setminus N\, \forall\, B \in \mathscr{B}(E) : P_{X|\mathscr{G}}(\omega, B) = P'_{X|\mathscr{G}}(\omega, B).$$

Proof See 44.3 Theorem in [Bau02]. □

Lemma B.20 *Let $(\Omega, \mathscr{F}, P)$ be some probability space, where Ω is a Polish space and where $\mathscr{F} := \mathscr{B}(\Omega)$. Furthermore, let $Y \in \mathscr{L}^1(\Omega)$ and let $\mathscr{G}$ be a sub–σ–algebra of $\mathscr{F}$. Then we have*

$$\forall\, A \in \mathscr{F} : \int_A Y(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, d\omega) = E(\mathbb{1}_A Y|\mathscr{G}) \quad P\text{–a.s.},$$

where $P_{\mathrm{id}\,|\mathscr{G}}$ denotes a regular $\mathscr{G}$–conditional distribution of id $: \Omega \to \Omega$ *(cf. Theorem B.19), and $\int_A Y(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, d\omega) : \Omega \to \mathbb{R}$ is given by $\int_A Y(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, d\omega)(\eta) = \int_A Y(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\eta, d\omega)$ for all $\eta \in \Omega$.*

Proof Fix some $A \in \mathscr{F}$. At first, consider a simple function $\varphi = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$ with $a_i \in \mathbb{R}_0^+$ and $A_i \in \mathscr{F}$ for $i = 1, \dots, n$. Then

$$\int_A \varphi(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, \mathrm{d}\omega) = \sum_{i=1}^n \alpha_i P_{\mathrm{id}\,|\mathscr{G}}(\cdot, A \cap A_i) = E\left(\mathbb{1}_A \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i} \middle| \mathscr{G}\right) = E(\mathbb{1}_A \varphi|\mathscr{G}) \quad P\text{–a.s..}$$

Now let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative simple function with $\varphi_n \uparrow Y^+$. Then we deduce from the above calculation that

$$\begin{aligned}
\int_A Y^+(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, \mathrm{d}\omega) &= \int_A \lim_{n\to\infty} \varphi_n(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, \mathrm{d}\omega) \\
&\overset{\text{DCT}}{=} \lim_{n\to\infty} \int_A \varphi_n(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, \mathrm{d}\omega) \\
&= \lim_{n\to\infty} E(\mathbb{1}_A \varphi_n|\mathscr{G}) \\
&\overset{\text{DCT}}{=} E\left(\mathbb{1}_A \lim_{n\to\infty} \varphi_n \middle| \mathscr{G}\right) \\
&= E(\mathbb{1}_A Y^+|\mathscr{G})
\end{aligned}$$

holds P–a.s.. Analogously, we obtain that $\int_A Y^-(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, \mathrm{d}\omega) = E(\mathbb{1}_A Y^-|\mathscr{G})$ holds true P–a.s., and thus

$$\begin{aligned}
\int_A Y(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, \mathrm{d}\omega) &= \int_A Y^+(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, \mathrm{d}\omega) - \int_A Y^-(\omega) P_{\mathrm{id}\,|\mathscr{G}}(\cdot, \mathrm{d}\omega) \\
&= E(\mathbb{1}_A Y^+|\mathscr{G}) - E(\mathbb{1}_A Y^-|\mathscr{G}) \\
&= E(\mathbb{1}_A Y|\mathscr{G})
\end{aligned}$$

holds P–a.s., which proves the assertion. □

Lemma B.21 *Let (E, ρ) be a metric space. Then for any nonempty and closed $A \subseteq E$ there exists a sequence $(f_n^A)_{n\in\mathbb{N}}$ of uniformly continuous functions $f_n^A : E \to [0, 1]$ which converges in a monotonically decreasing manner pointwise to $\mathbb{1}_A$ as $n \to \infty$.*

Proof Let $\varphi : \mathbb{R} \to [0, 1]$ be defined by

$$\begin{cases} 1, & t < 0 \\ 1 - t, & 0 \le t \le 1 \\ 0, & t > 1. \end{cases}$$

Let $\emptyset \neq A \subseteq E$ be closed and consider $d_A : E \to \mathbb{R}_0^+$, $x \mapsto \inf\{\rho(x, y) : y \in A\}$, the distance between some $x \in E$ and A, and $f_n^A : E \to [0, 1]$, $n \in \mathbb{N}$, defined by $f_n^A(x) = \varphi(n \cdot d_A(x))$.

We have to show that d_A is uniformly continuous. This can be seen as follows: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E converging to $x \in E$, $\varepsilon > 0$, $\delta := \varepsilon$, and let $n \in \mathbb{N}$ be sufficiently large so that $\rho(x_n, x) < \delta$. Then

$$|d_A(x_n) - d_A(x)| = \left|\inf_{y\in A} \rho(x_n, y) - \inf_{y\in A} \rho(x, y)\right| \leq \sup_{y\in A} |\rho(x_n, y) - \rho(x, y)| \overset{(*)}{\leq} \rho(x_n, x) < \delta = \varepsilon,$$

where $(*)$ holds because $\rho(\cdot, y)$ is uniformly continuous for all $y \in E$.

Therefore, every f_n^A, $n \in \mathbb{N}$, is uniformly continuous as a thesis of uniformly continuous functions.

Let $x \in E$ and firstly assume $d_A(x) > 0$. Then $x \notin A$ and hence

$$|f_n^A(x) - \mathbb{1}_A(x)| = f_n^A(x) = \varphi(n \cdot d_A(x)) \downarrow 0 \text{ as } n \to \infty.$$

In case that $d_A(x) = 0$ we have that $x \in A$ and hence

$$\forall\, n \in \mathbb{N} : |f_n^A(x) - \mathbb{1}_A(x)| = |1 - 1| = 0.$$

□

Theorem B.22 *Let E be a locally compact Polish space, and let T be the generator of a Feller semigroup $(T_t)_{t\in\mathbb{R}_0^+}$ on $C_0(E)$. Furthermore, let $\mathcal{D}$ denote the domain of T.*

(i) Then we have

$$\forall\, f \in \mathcal{D}\, \forall\, t \in \mathbb{R}_0^+ : T_t f - f = \int_{[0,t]} (T_s \circ T) f \lambda(ds). \tag{B.10}$$

(ii) $T_{(\cdot)} f$ is differentiable at 0 iff $f \in \mathcal{D}$, and if one of these properties holds, then $T_{(\cdot)}$ is differentiable on $\mathbb{R}_0^+$ and

$$\forall\, f \in \mathcal{D}\, \forall\, t \in \mathbb{R}_0^+ : \frac{d}{dt}(T_t f) = T_t T f = T T_t f. \tag{B.11}$$

Proof Confer Lemma 17.6 in [Kal01]. □

Theorem B.23 *Let E be a locally compact Polish space, and let T be the generator of a Feller semigroup $(T_t)_{t\in\mathbb{R}_0^+}$ on $C_0(E)$. Then T satisfies the positive maximum principle (cf. Definition A.11).*

Proof Let $\mathcal{D}$ denote the domain of T. We firstly note that T_t, $t \in \mathbb{R}_0^+$, is monotonically increasing on $C_0(E)$, i.e., $f \leq g \in C_0(E) \implies T_t f \leq T_t g$. This assertion holds, since T_t, $t \in \mathbb{R}_0^+$ is a positive linear operator. Choose $f \in \mathcal{D}$ and $x_0 \in E$ such that $f^+ \leq f(x_0)$. Using that T_t, $t \in \mathbb{R}_0^+$, is a contraction operator we obtain

$$\forall\, t \in \mathbb{R}_0^+ : T_t f(x_0) \leq T_t f^+(x_0) \leq \|T_t f^+\|_\infty \leq \|f^+\|_\infty = f(x_0).$$

Hence $h^{-1}(T_h f - f)(x_0) \leq 0$ for all $h \in 0$ and thus we get by Theorem B.22 (ii)

$$Tf(x_0) = \lim_{h \to 0} \frac{(T_h f - f)(x_0)}{h} \leq 0.$$

□

In conclusion we present a result from topology which is indispensable for our considerations.

Theorem B.24 *Let E be a locally compact topological space. Then there exists a uniquely (up to homeomorphisms) defined compact Hausdorff space $\hat{E}$ which contains a space $\tilde{E}$, which is homeomorphic to E, such that there exists some Δ with $\hat{E} \backslash \tilde{E} = \{\Delta\}$. If E is not compact, then $\tilde{E}$ is dense in $\hat{E}$. The space $\hat{E}$ is called* **Alexandroff one–point compactification** *of E.*

Proof See 8.18 Theorem in [Que01]. □

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Index