Killed fragmentations and intrinsic spectrally negative Lévy processes

Robert Knobloch, Department of Mathematical Sciences, University of Bath, Bath, UK

Homogenous fragmentation processes

Let $\mathcal{P}$ denote the space of of partitions $\pi := (\pi_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$, ordered such that

$$\forall\, i \le j \in \mathbb{N}: \quad \inf \pi_i \le \inf \pi_j,$$

where $\inf \emptyset := \infty$.

Definition *We call a $\mathcal{P}$–valued Markov process $\Pi := (\Pi(t))_{t\in\mathbb{R}_0^+}$, being continuous in probability, homogenous $\mathcal{P}$–fragmentation process if*

- $\Pi(0) = (\mathbb{N}, \emptyset, \ldots)$.
- *For any $s, t \in \mathbb{R}_0^+$, given that $\Pi(t) = (\pi_n)_{n\in\mathbb{N}}$, we have*

$$\Pi(s+t) \stackrel{\mathrm{d}}{=} \left(\pi_n \cap \Pi^{(n)}(s)\right)_{n\in\mathbb{N}},$$

reordered to be an element of $\mathcal{P}$, where the $\Pi^{(n)}$ are i.i.d. copies of Π.

Let $\mathscr{F} := (\mathscr{F}_t)_{t\in\mathbb{R}_0^+}$ denote the filtration generated by the process Π.

In [Ber01] it is shown that the blocks of Π have *asymptotic frequencies*, that is

$$|\Pi_n(t)| := \lim_{k\to\infty} \frac{\mathrm{card}(\Pi_n(t) \cap \{1, \ldots, k\})}{k}$$

exist $\mathbb{P}$–a.s. for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$.

Set

$$\mathcal{S} := \left\{ \mathbf{s} := (s_n)_{n\in\mathbb{N}} \subseteq [0,1] : \sum_{n\in\mathbb{N}} s_n \le 1,\ s_i \ge s_j\ \forall\, i \le j \right\}$$

and consider a measure ν, called *dislocation measure*, on $\mathcal{S}$ that satisfies the following conditions:

$$\int_{\mathcal{S}} (1 - s_1)\nu(\mathrm{d}\mathbf{s}) < \infty \qquad \text{and} \qquad \nu(\{a, 0, \ldots\}) = 0 \quad \forall\, a \in [0,1].$$

Further, define

$$\underline{p} := \inf\left\{ p \in \mathbb{R} : \int_{\mathcal{S}} \left| 1 - \sum_{n\in\mathbb{N}} s_n^{1+p} \right| \nu(\mathrm{d}\mathbf{s}) < \infty \right\} \in (-1, 0].$$

It is well known that

$$\Phi : (\underline{p}, \infty) \to \mathbb{R}, \quad p \mapsto \Phi(p) = \int_{\mathcal{S}} \left(1 - \sum_{n\in\mathbb{N}} s_n^{1+p} \right) \nu(\mathrm{d}\mathbf{s})$$

is strictly monotonically increasing and concave.

Bertoin [Ber01] showed that the process $(-\ln(|\Pi_1(t)|))_{t\in\mathbb{R}_0^+}$ is a killed subordinator with Laplace exponent Φ. Its killing rate ζ is exponentially distributed with parameter $\Phi(0)$. Hence,

$$\forall\, p \in (\underline{p}, \infty): \quad \Phi(p) = -\frac{1}{t} \ln\left(\mathbb{E}\left(e^{p \ln(|\Pi_1(t)|)} \mathbb{1}_{\{t<\zeta\}} \right) \right) = -\frac{1}{t} \ln\left(\mathbb{E}\left(|\Pi_1(t)|^p \mathbb{1}_{\{t<\zeta\}} \right) \right).$$

In view of [Ber03] let $\bar{p}$ be the unique solution to

$$(1+p)\Phi'(p) = \Phi(p),$$

where Φ' denotes the derivative of Φ. Define

$$c_{\bar{p}} := \Phi'(\bar{p}) = \frac{\Phi(\bar{p})}{1+\bar{p}}.$$

Intrinsic killed spectrally negative Lévy processes

For any $t \in \mathbb{R}_0^+$ let $B_n(t)$ denote the block of $\Pi(t)$ that contains the element $n \in \mathbb{N}$.

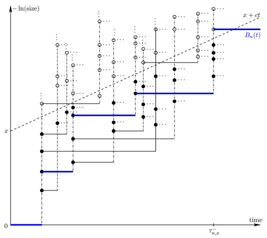

Figure 1: Illustration of $(|B_n(t)|)_{t\in\mathbb{R}_0^+}$ incl. the killing line with slope $c > 0$ starting at $x \in \mathbb{R}_0^+$. The black dots show particles alive in the killed process as their paths are below the killing line.

For all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0^+$ set

$$\tau_{n,x}^- := \inf\{t \in \mathbb{R}_0^+ : -\ln(B_n(t)) > x + ct\}.$$

For any $n \in \mathbb{N}$ and $x \in \mathbb{R}_0^+$ consider the process $X_n^x := (X_n^x(t))_{t\in\mathbb{R}_0^+}$ given by

$$\forall\, t \in \mathbb{R}_0^+: \quad X_n^x(t) := (x + ct + \ln(|B_n(t)|))\, \mathbb{1}_{\{\tau_{n,x}^- > t\}}.$$

The process X_n^x is a spectrally negative Lévy process shifted by x and killed on hitting the interval $(-\infty, 0)$. We denote the unkilled version of this process by X_n, i.e. $X_n(t) = X_n^x(t)$ for all $t < \tau_{n,x}^-$.

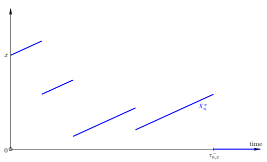

Figure 2: Illustration of X_n^x. Between the jumps the slope c of X_n^x is the same c as in Figure 1. At time $\tau_{n,x}^-$ the unkilled process X_n hits the interval $(-\infty, 0)$, thus $X_n^x(t) = 0$ for all $t \ge \tau_{n,x}^-$.

A block $\Pi_n(t)$ of Π is killed at the moment $t \in \mathbb{R}_0^+$ of its creation if

$$|\Pi_n(t)| < e^{-(x+ct)}.$$

A block that is killed is set to be $(0, \ldots) \in \mathcal{S}$ and the killed process is denoted by $\Pi^x := (\Pi_n^x)_{n\in\mathbb{N}}$.

For any $x \in \mathbb{R}_0^+$ the, not necessarily finite, extinction time of Π^x is given by

$$\zeta^x := \sup_{n\in\mathbb{N}} \tau_{n,x}^-.$$

For all $n \in \mathbb{N}$ and $t, x \in \mathbb{R}_0^+$ set

$$\kappa_{x,n,t} := \inf \Pi_n^x(t) \qquad \text{as well as} \qquad \mathcal{N}_t^x := \left\{ k \in \mathbb{N} : t < \tau_{\kappa_{x,k,t},x}^- \right\}.$$

That is, $\mathcal{N}_t^x$ consists of all the indices of blocks that are not yet killed by time t.

For any $t, x \in \mathbb{R}_0^+$ define

$$\lambda_1^x(t) := \sup_{n\in\mathbb{N}} |\Pi_n^x(t)|.$$

Note that $\lambda_1^x(t) = 0$ for all $t \ge \zeta^x$.

Main results

Proposition *If $c \le c_{\bar{p}}$, then $\mathbb{P}(\zeta^x < \infty) = 1$ for all $x \in \mathbb{R}_0^+$. If $c > c_{\bar{p}}$, then*

$$x \mapsto \mathbb{P}(\zeta^x < \infty)$$

is a continuous and strictly monotonically decreasing $(0,1)$–valued function on $\mathbb{R}_0^+$.

Observe that

$$\forall\, t, x \in \mathbb{R}_0^+\ \forall\, n \in \mathcal{N}_t^x: \quad X_{\kappa_{x,n,t}}^x(t) = x + ct + \ln(|\Pi_n^x(t)|).$$

For $p \in (\underline{p}, \infty)$ define a stochastic process $M^x(p) := (M_t^x(p))_{t\in\mathbb{R}_0^+}$ by

$$M_t^x(p) := \sum_{n\in\mathcal{N}_t^x} W_p\left(X_{\kappa_{x,n,t}}^x(t) \right) e^{\Phi(p)t} |\Pi_n^x(t)|^{1+p},$$

where W_p is the scale function of X_1 under the changed measure $\mathbb{P}^{(p)}$ given by

$$\forall\, t \in \mathbb{R}_0^+: \quad \left.\frac{\mathrm{d}\mathbb{P}^{(p)}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = e^{\Phi(p)t + p\ln(|B_1(t)|)}.$$

.

Theorem *Let $c > c_{\bar{p}}$ and let $p \in (\underline{p}, \bar{p})$ be such that $c > \Phi'(p)$. Then the process $M^x(p)$ is a nonnegative $\mathscr{F}$–martingale with $\mathbb{P}$–a.s. limit $M_\infty^x(p)$ that satisfies*

$$\mathbb{P}(\{M_\infty^x(p) = 0\} \triangle \{\zeta^x < \infty\}) = 0,$$

where $\triangle$ denotes the symmetric difference.

For any function $f : \mathbb{R}^+ \to [0,1]$ let $Z^{x,f} := (Z_t^{x,f})_{t\in\mathbb{R}_0^+}$ be given by

$$Z_t^{x,f} = \prod_{n\in\mathcal{N}_t^x} f\left(X_{\kappa_{x,n,t}}^x(t) \right).$$

Theorem *Let $c > c_{\bar{p}}$. Then there exists a unique monotone function $f : \mathbb{R}_0^+ \to [0,1]$, given by*

$$\forall\, x \in \mathbb{R}_0^+: \quad f(x) = \mathbb{P}(\zeta^x < \infty),$$

for which $Z^{x,f}$ is an $\mathscr{F}$–martingale for any $x \in \mathbb{R}_0^+$ and that satisfies $\lim_{x\to\infty} f(x) = 0$.

Proposition *Let $c > c_{\bar{p}}$ and $x \in \mathbb{R}_0^+$. Then we have*

$$\lim_{t\to\infty} \frac{-\ln(\lambda_1^x(t))}{t} = c_{\bar{p}}$$

$\mathbb{P}(\cdot\,|\,\zeta^x = \infty)$–almost surely.

Concluding remark In a forthcoming paper we use our results on killed fragmentation processes in order to obtain existence– and uniqueness results for one–sided FKPP travelling waves in the setting of fragmentation processes.

References

[Ber01] J. Bertoin. Homogeneous fragmentation processes, *Probab. Theory Related Fields* **121**, pp. 301–318, 2001

[Ber03] J. Bertoin. The asymptotic behavior of fragmentation processes, *J. Europ. Math. Soc.*, **5**, pp. 395–416, 2003

E–mail: r.knobloch@bath.ac.uk

Web page: http://www.maths.bath.ac.uk/~rk243