An Itō formula for convoluted Lévy processes

Robert Knobloch, joint work with Christian Bender and Philip Oberacker
Department of Mathematics, Saarland University, Germany

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Convoluted Lévy processes

Let $(\tilde{L}(t))_{t\in\mathbb{R}_0^+}$ be a Lévy process with jump measure N and characteristic triplet (γ,σ,ν) such that $\mathbb{E}(\tilde{L}(1)) = 0$ and $\tilde{L}(t) \in \mathscr{L}^k(\mathbb{P})$ for all $k \in \mathbb{N}$. We construct a two-sided Lévy process $L := (L(t))_{t\in\mathbb{R}}$ by taking two independet copies $(L_1(t))_{t\in\mathbb{R}_0^+}$ and $(L_2(t))_{t\in\mathbb{R}_0^+}$ of $\tilde{L}$ and defining

$$L(t) := \begin{cases} L_1(t), & t \geq 0 \\ -L_2(-t-), & t < 0. \end{cases}$$

In what follows, λ denotes the Lebesgue measure on $\mathbb{R}$. We are interested in stochastic integrals with respect to L of kernels in the following class $\mathcal{K}$ of measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$ which satisfy

- $\forall\, s > t \geq 0 :\ f(t,s) = 0,$
- $f(0,\cdot) = 0 \quad \lambda$-a.e.,
- $\forall\, k \geq 2, \forall\, t \in \mathbb{R} :\ f(t,\cdot) \in \mathscr{L}^k(\lambda),$
- certain regularity conditions on $[0,b) \times \mathbb{R}$ for some $b \in (0,\infty]$,
- $\lambda\,(s \in \mathbb{R} : f(t,s) \neq 0) \neq 0$ for λ-a.a. $t \in [0,b)$.

Definition *For any $f \in \mathcal{K}$ we define a stochastic process $M := (M(t))_{t\in\mathbb{R}}$ by*

$$M(t) = \int_{\mathbb{R}} f(t,s)L(\mathrm{d}s)$$

for every $t \in \mathbb{R}$. The process M is referred to as **convoluted Lévy process**.

Since $f(t,\cdot) \in \mathscr{L}^2(\lambda)$, the integral in the above definition exists as an element of $\mathscr{L}^2(\mathbb{P})$ according to the theory of stochastic integration with respect to semimartingales.

Examples *Examples of convoluted Lévy processes M with $f \in \mathcal{K}$ include*

- **one-sided shot noise processes** *with kernel*

$$f(t,s) = \begin{cases} k(t-s), & 0 \leq s \leq t \leq b \\ 0, & \text{otherwise} \end{cases}$$

for some $k \in \mathbb{N}$ and $b \in \mathbb{R}^+$,

- **one-sided Ornstein-Uhlenbeck type processes** *with kernel*

$$f(t,s) = \begin{cases} e^{-k(t-s)}, & 0 \leq s \leq t \leq b \\ 0, & \text{otherwise} \end{cases}$$

for some $k \in \mathbb{N}_0$ and $b \in \mathbb{R}^+$,

- **fractional Lévy processes** *with the Mandelbrot-van Ness kernel*

$$f_d(t,s) = \frac{1}{\Gamma(d+1)}\left((t-s)_+^d - (-s)_+^d\right)$$

for all $s,t \in \mathbb{R}$ and a fractional integration parameter $d \in (0, 1/2)$.
Notation: $M^d := M$ with kernel f_d.

In particular, the fractional Lévy processes M^d, $d \in (0,1/2)$, have the following properties:

- M^d has a modification which is pathwise defined as an improper Riemann integral and its paths are $\mathbb{P}$-a.s. locally Hölder continuous of every order $\alpha < d$.
- M^d has the same correlation structure as fractional Brownian motion with $H = d + 1/2$.
- M^d exhibits long-range dependence and has stationary increments.
- $M^d(t)$ is infinitely divisible for every $t \in \mathbb{R}$.
- M^d is not a martingale and for a large class of Lévy processes it is not a semimartingale either.
- M^d is not self-similar and is not a Markov process.

Hitsuda-Skorokhod integrals

Let I_n be the n-th order multiple Lévy-Itō integral. For any $g \in \mathscr{L}^2(x^2\nu(\mathrm{d}x) \times \lambda(\mathrm{d}t))$ let $g^{\otimes n}$ denote the n-fold tensor product of g and define a measure $\mathcal{Q}_g$ on $(\Omega,\mathbb{P})$ by the change of measure

$$\mathrm{d}\mathcal{Q}_g = \sum_{n=0}^{\infty} \frac{I_n(g^{\otimes n})}{n!}\,\mathrm{d}\mathbb{P},$$

where the Radon-Nikodým derivative is the Wick exponential of $I_1(g)$.

A crucial tool for our considerations is the $\mathcal{S}$-transform as given by the following definition:

Definition *For every $\varphi \in \mathscr{L}^2(\mathbb{P})$ we define its $\mathcal{S}$-transform $\mathcal{S}(\varphi)$ by*

$$\forall\, g \in \mathscr{L}^2(x^2\nu(\mathrm{d}x) \times \lambda(\mathrm{d}t)) : \quad \mathcal{S}(\varphi)(g) := \mathbb{E}^{\mathcal{Q}_g}(\varphi),$$

where $\mathbb{E}^{\mathcal{Q}_g}$ denotes the expectation under the probability measure $\mathcal{Q}_g$.

With $g^*(x,t) := xg(x,t)$ for all $(x,t) \in \mathbb{R}^2$ let us introduce a set Ξ by

$$\Xi := \operatorname{span}\left\{g \in \mathscr{L}^2(x^2\nu(\mathrm{d}x) \times \lambda(\mathrm{d}t)) : g^* \in \mathscr{L}^1(\nu \otimes \lambda); g(x,t) = g_1(x)g_2(t) : g_1 \in \mathscr{L}^\infty \wedge g_2 \in \mathscr{S}\right\},$$

where $\mathscr{L}^\infty$ and $\mathscr{S}$ denote the sets of essentially bounded functions and Schwartz functions on $\mathbb{R}$.

In order to define Hitsuda-Skorokhod integrals we shall need the following injectivity property of $\mathcal{S}$:

Proposition 1 *Let φ, ψ be in $\mathscr{L}^2(\mathbb{P})$. If $\mathcal{S}(\varphi)(g) = \mathcal{S}(\psi)(g)$ for all $g \in \Xi$, then we have $\varphi = \psi$ $\mathbb{P}$-almost surely.*

Based on Proposition 1 we define Hitsuda-Skorokhod integrals with respect to the jump measure N.

Definition *Let $B \subseteq \mathbb{R}$ be some Borel set and let $X : \mathbb{R} \times B \times \Omega \to \mathbb{R}$ be a random field. The* **Hitsuda-Skorokhod integral** *of X with respect to N is said to exist in $\mathscr{L}^2(\mathbb{P})$, if there is some $\Phi \in \mathscr{L}^2(\mathbb{P})$ that satisfies*

$$\mathcal{S}(\Phi)(g) = \int_B \int_{\mathbb{R}} \mathcal{S}(X(x,t))(g)(g^*(x,t)+1)\nu(\mathrm{d}x)\mathrm{d}t$$

for every $g \in \Xi$. Notation:

$$\int_B \int_{\mathbb{R}} X(y,t)N^\diamond(dy,\mathrm{d}t) := \Phi.$$

If the integrand is predictable, then the above Hitsuda-Skorokhod integral is just an ordinary stochastic integral with respect to a random measure. The following definition is based on a similar motivation.

Definition *Suppose the map $t \mapsto S(M(t))(g)$ is differentiable for every $g \in \Xi$, $B \subseteq \mathbb{R}$ is a Borel set and $X : B \times \Omega \to \mathbb{R}$ is a stochastic process. Then X has a* **Hitsuda-Skorokhod integral** *with respect to M, if there exists a random variable $\Phi \in \mathscr{L}^2(\mathbb{P})$ such that*

$$\mathcal{S}(\Phi)(g) = \int_B S(X(t))(g)\frac{d}{dt}\mathcal{S}(M(t))(g)\mathrm{d}t$$

holds for all $g \in \Xi$. Notation:

$$\int_B X(t)M^\diamond(\mathrm{d}t) := \Phi.$$

Main result

Our main result is the following Itō type formula:

Theorem 1 [Bender, K., Oberacker 2013] *Let $\sigma > 0$, $f \in \mathcal{K}$, $T \in [0,b] \cap \mathbb{R}_0^+$ and let $G \in C^2(\mathbb{R})$ be such that G, G' and G'' are of polynomial growth. Then*

$$\begin{aligned} G(M(T)) = G(0) &+ \sigma^2 \int_0^T G''(M(t-))\left(\frac{f(t,t)^2}{2} + \int_{-\infty}^t f(t,s)\frac{\partial}{\partial t}f(t,s)\mathrm{d}s\right)\mathrm{d}t \\ &+ \sum_{0<t\leq T}\left(G(M(t)) - G(M(t-)) - G'(M(t-))\Delta M(t)\right) \qquad (1) \\ &+ \int_0^T \int_{-\infty}^t \int_{\mathbb{R}} \left(G'(M(t-)+xf(t,s)) - G'(M(t-))\right) x\frac{\partial}{\partial t}f(t,s)N^\diamond(\mathrm{d}x,\mathrm{d}s)\mathrm{d}t \\ &+ \int_0^T G'(M(t-))M^\diamond(\mathrm{d}t) \end{aligned}$$

holds $\mathbb{P}$-a.s., if all the terms exist in $\mathscr{L}^2(\mathbb{P})$.

The reason why we need that L has a nontrivial Gaussian component is the following result:

Proposition 2 *Let $f \in \mathcal{K}$ and let $t \in \mathbb{R}$ be such that $f(t,\cdot)$ does not vanish λ-a.e.. If $\sigma > 0$, then $u \mapsto \mathbb{E}^{\mathcal{Q}_g}(e^{iuM(t)})$ is a Schwartz function on $\mathbb{R}$.*

The underlying idea of the proof of Theorem 1 is to show that the S-transforms of the LHS and RHS in (1) coincide and subsequently to resort to Proposition 1. Our starting point is

$$\mathcal{S}(G(M(T)))(g) = G(0) + \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(G(M(t)))(g)\mathrm{d}t.$$

Let $\mathcal{F}$ denote the Fourier transform. By means of the Fourier inversion theorem we obtain

$$\mathcal{S}(G(M(t)))(g) = \mathbb{E}^{\mathcal{Q}_g}(G(M(t))) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} G(u)\mathcal{F}\left(\mathbb{E}^{\mathcal{Q}_g}\left(e^{i(\cdot)M(t)}\right)\right)(u)\mathrm{d}u \qquad (2)$$

for any $g \in \Xi$, where according to Proposition 2 the integral on the RHS of (2) exists λ-a.e. on $[0,b)$. In this spirit, the proof of Theorem 1 is based on the following representation of $\mathbb{E}^{\mathcal{Q}_g}(e^{iuM(t)})$, which in particular enables us to compute the derivative of the RHS in (2).

Lemma *For all $f \in \mathcal{K}$, $g \in \Xi$, $t \in \mathbb{R}_0^+$ and $u \in \mathbb{R}$ we have*

$$\begin{aligned} &\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) \\ &= \exp\Bigg(iu\int_{-\infty}^t \sigma^2 f(t,s)g(0,s)\mathrm{d}s + iu\int_{-\infty}^t\int_{\mathbb{R}} xf(t,s)g^*(x,s)\nu(\mathrm{d}x)\mathrm{d}s \\ &\quad -\frac{\sigma^2u^2}{2}\int_{-\infty}^t f(t,s)^2\mathrm{d}s + \int_{-\infty}^t\int_{\mathbb{R}}\left(e^{iuxf(t,s)} - 1 - iuxf(t,s)\right)(1+g^*(x,s))\nu(\mathrm{d}x)\mathrm{d}s\Bigg). \end{aligned}$$

Concluding remark The assumption of Theorem 1 that all the terms appearing in (1) exist in $\mathscr{L}^2(\mathbb{P})$ is rather restrictive. Therefore, our current research is focused on providing sufficient criteria in the Lévy white noise framework under which those terms exist as generalised random variables. However, the only terms on the RHS of (1) which may not exist separately are the last two terms, which always exist as a combined object in $\mathscr{L}^2(\mathbb{P})$.

E-mail: knobloch@math.uni-sb.de
Web page: http://www.math.uni-sb.de/ag-bender/knobloch.html

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