Simple Arbitrage

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Abstract

We characterize absence of arbitrage with simple trading strategies in a discounted market with a constant bond and a stock. We show that, if there is a simple arbitrage, then there is a 0-admissible one or an obvious one, i.e. a simple arbitrage which promises a minimal risk-less gain of $\epsilon$, if the investor trades at all. For continuous stock models we provide an equivalent condition for absence of 0-admissible simple arbitrage in terms of a property of the fine structure of the paths, which we call ‘infinitesimal up’n’down’. This property can be verified for many models by the law of the iterated logarithm. As an application we show that the mixed fractional Black-Scholes model with Hurst parameter bigger than a half is free of simple arbitrage on a compact time horizon. More generally, we discuss absence of simple arbitrage for stochastic volatility models and local volatility models which are perturbed by an independent 1/2-Hölder continuous process.

Keywords: arbitrage, conditional full support, fractional Brownian motion, law of the iterated logarithm, simple strategies

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1 Introduction

The fundamental theorem of asset pricing characterizes absence of arbitrage in terms of the existence of equivalent martingale measures. More precisely, the version of the fundamental theorem obtained by Delbaen and Schachermayer [10] states that a locally bounded stock model does not admit a free lunch with vanishing risk, if and only if the the model has an equivalent local martingale measure. Clearly, absence of arbitrage and related notions such as no free lunch with vanishing risk heavily depend on the class of admissible strategies. The fundamental theorem in [10] utilizes the class of nds-admissible strategies. This class includes the self-financing strategies whose discounted wealth process is bounded from below. This boundedness assumption rules out so-called doubling strategies. (Note that

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nds stands for ‘no-doubling strategy’). It is well-known that without the nds-condition there are self-financing arbitrage opportunities even in the classical Black-Scholes model (see e.g. [4] for some explicit examples). As absence of arbitrage is generally considered as a minimum requirement for a sensible stock model, non-semimartingale models are ruled out in the setting of the fundamental theorem.

More recently, several authors studied no arbitrage problems beyond the semimartingale setting by either introducing market imperfections such as transaction costs [14, 15, 16] or by restricting the class of admissible strategies [3, 4, 7, 18]. Under proportional transaction costs Guasoni et al. [16] derive a version of the fundamental theorem, which characterizes absence of arbitrage in terms of the existence of $\epsilon$-consistent price systems. A sufficient condition for the latter is conditional full support [15], which is e.g. satisfied by a fractional Brownian motion and many other non-semimartingale models.

Without imposing transaction costs, absence of arbitrage can be obtained in some non-semimartingale models by restricting the class of admissible strategies. Bender et al. [3] show absence of arbitrage for a class of perturbed local volatility models, including the mixed fractional Black-Scholes model with Hurst parameter $H > 1/2$, with a class of strategies which covers smooth functions of time, the spot price, the running minimum and maximum of the stock price and the running average of the stock price. Cheridito [7] introduces a class of strategies where the portfolio can only be readjusted at finitely many stopping times and there must be a minimum time between two readjustments. This class of strategies was called Cheridito class by Jarrow et al. [18]. Cheridito proves absence of arbitrage with this class of strategies for fractional Brownian motion with arbitrary Hurst parameter. Jarrow et al. [18] show that absence of arbitrage in the Cheridito class holds for models which can be written as a sum of a continuous local martingale and an RCLL bounded process under some conditions on the quadratic variation of the local martingale. Bender et al. [4] note that conditional full support is sufficient for absence of arbitrage in the Cheridito class and even for a larger class of strategies where the minimum time between readjustments is localized in a suitable way.

The aim of the present paper is to characterize absence of simple arbitrage (i.e. absence of arbitrage with simple strategies) and to provide some classes of models which are free of simple arbitrage and include non-semimartingales. We recall here, that a strategy is simple, if the portfolio is readjusted at $n$ stopping times for some positive integer $n$. Due to the bounded number of readjustment times, a simple arbitrage can never be obtained by a doubling scheme and therefore, as usually, we do not impose the nds-boundedness condition on simple strategies.

It is well-known that the existence of an equivalent local martingale measure is neither necessary nor sufficient for absence of simple arbitrage.
Delbaen and Schachermayer [11] provide an example of a strict local martingale on a compact time interval which admits simple arbitrage. In their example, the local martingale has initial value 1 and terminal value 0, and so an arbitrage can be obtained by going short in the stock (cp. also the related Example 2.2, (i), below). Recently, Bayraktar and Sayit [2] provide an equivalent condition for a non-negative strict local martingale to be free of simple arbitrage on a compact time horizon. In the terminology, which we use in the present paper, the main result in [2] states that absence of simple arbitrage for an RCLL non-negative strict local martingale on a compact time horizon is equivalent to absence of an obvious arbitrage by going short in the stock. Here an obvious arbitrage is a strategy of the form $\pm 1_{(\sigma, \tau]}$ and if the investor trades at all, i.e. on the set $\{\sigma < \tau\}$, she can be sure to have a riskless gain of at least a given constant $\epsilon > 0$, compare Definition 2.3 below. Corollary 2.8 below can be considered as a generalization of the main result in [2] to an unbounded time horizon.

A classical example of a process, which does not have an equivalent local martingale measure and is, nonetheless, free of simple arbitrage on a compact time interval, is $W_t + \frac{t^{1/2}}{2}$ for a Brownian motion $W$. This example is due to Delbaen and Schachermayer [12] and is also discussed by Jarrow et al. [18]. Corollary 6.2 below shows that absence of simple arbitrage on a compact time horizon extends to models of the form $W_t + Y_t$, where $Y$ is 1/2-Hölder continuous and independent of the Brownian motion $W$. This result covers the mixed fractional Black-Scholes model with Hurst parameter $H > 1/2$, which is known not to be a semimartingale for the range $1/2 < H \leq 3/4$, [6]. We also discuss absence of arbitrage for ‘mixed’ stochastic volatility models and ‘mixed’ local volatility models (where ‘mixed’ refers to adding an 1/2-Hölder continuous process to the log-price of the original model).

The paper is organized as follows. In Section 2 we provide a characterization of simple arbitrage for right-continuous discounted stock models. We show that existence of a simple arbitrage implies existence of an obvious arbitrage or existence of a 0-admissible arbitrage. These two types of arbitrages are particularly favourable for an investor: Obvious arbitrages guarantee a minimum riskless gain, if the investor starts to trade at all; 0-admissible arbitrages can be obtained without running into debt while waiting for the riskless gain. In Section 3 we specialize to models with continuous paths. For such models we give an equivalent condition for the absence of 0-admissible arbitrage in terms of a property on the fine structure of the paths, which we call ‘infinitesimal up’n’down’. This property roughly says: Given a stopping time $\sigma$, as soon as the stock price $X_t$, $t \geq \sigma$, moves from the level $X_\sigma$, it will cross $X_\sigma$ infinitely often in time intervals of length $\epsilon$ for every $\epsilon > 0$. The main result in Section 4 provides a sufficient condition for ‘infinitesimal up’n’down’ and, thus, for absence of simple 0-admissible arbitrage. We prove that this property holds for processes of the form $M_t + Y_t$, where $M$ is a continuous local martingale and $Y$ is 1/2-Hölder continuous with respect
to the quadratic variation of $M$. The proof makes use of the law of the iterated logarithm for a Brownian motion and a time change argument. In Section 5 we have a closer look at obvious arbitrage and recall some facts on the conditional full support property, which turns out to be a sufficient condition for absence of obvious arbitrage on compact time intervals. Section 6 illustrates the results with several examples.

2 A characterization of simple arbitrage for right-continuous processes

In this section we provide a characterization of simple arbitrage. We assume that a discounted market with two securities is given. A constant bond $B_t = 1$ and a stock modelled by a right-continuous adapted stochastic process $X_t$, $t \in [0, \infty)$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, P)$. The filtered probability space is assumed to satisfy the usual conditions of completeness and right-continuity of the filtration.

An investor can trade into the market by choosing the number of stocks held at time $t$ by a simple strategy of the form

$$\Phi_t = \phi_0 1_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j 1_{(\tau_j, \tau_{j+1}]}(t),$$

where $n \in \mathbb{N}$, $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n$ are a.s. finite stopping times with respect to $(\mathcal{F}_t)$ and the $\phi_j$ are $\mathcal{F}_{\tau_j}$-measurable real random variables. Note that the trader is allowed to trade on an infinite time horizon, because we do not restrict to bounded stopping times for the re-allocation of the capital. Of course, trading on a finite time horizon $[0, T]$ is covered by switching to the process $(X_{t \wedge T}, \mathcal{F}_{t \wedge T})$.

As the market is already discounted, the self-financing condition on the simple strategy $\Phi$ enforces that the investor’s wealth at time $t \in [0, \infty)$ is given by

$$V_t(\Phi; v) = v + \sum_{j=0}^{n-1} \phi_{j+1}(X_{t \wedge \tau_{j+1}} - X_{t \wedge \tau_j}),$$

where $v$ is the investor’s initial capital. The wealth process $V_t(\Phi; v)$ inherits right-continuity from $X$ and satisfies

$$V_\infty(\Phi; v) = \lim_{t \to \infty} V_t(\Phi; v) = v + \sum_{j=0}^{n-1} \phi_{j+1}(X_{\tau_{j+1}} - X_{\tau_j}),$$

because the stopping times $\tau_j$, $j = 1, \ldots, n$, are finite $P$-almost surely.

Definition 2.1. A simple strategy $\Phi$ is
• an arbitrage, if $V_\infty(\Phi; 0) \geq 0$ $P$-a.s. and $P(\{V_\infty(\Phi; 0) > 0\}) > 0$. 
• $c$-admissible for some $c \geq 0$, if 
  \[ \inf_{t \in [0, \infty)} V_t(\Phi; 0) \geq -c \quad P\text{-almost surely.} \]

We will speak of a simple arbitrage $\Phi$, if $\Phi$ is a simple strategy and an arbitrage.

We will illustrate two types of simple arbitrages, which will play an important role throughout this paper, by examples.

**Example 2.2.** (i) Suppose $W_t$ a Brownian motion and for some fixed $T > 0$

\[ X_t = \begin{cases} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{w^2}{2(T-t)}}, & 0 \leq t < T \\ 0, & t \geq T \end{cases} \]

Then, $X_t$ has continuous paths $P$-almost surely and is a local martingale, which can be easily verified by an application of Itô’s formula. As $X_0 = \frac{1}{\sqrt{2\pi T}} > 0$ and $X_T = 0$, we observe that the simple strategy $\Phi_t = -1_{(0,T]}(t)$ is an arbitrage.

Here the arbitrage is obtained in the ‘long run’ by waiting up to time $T$. Borrowing the terminology of Guasoni et al. [16] this arbitrage is an obvious arbitrage. This means that the arbitrage is of the form $\pm 1_{[\sigma, \tau]}$ and if the investor trades at all, i.e. on the set $\{\sigma < \tau\}$, she can be sure to have a riskless gain of at least a given constant $\epsilon > 0$ (here: $\frac{1}{\sqrt{2\pi T}}$), compare Definition 2.3 below. Note that in the present example there is no $c \geq 0$ such that the arbitrage is $c$-admissible.

(ii) Suppose $X_t = \exp\{W_t + t^\alpha\}$ for some $\alpha < 1/2$. By the law of the iterated logarithm we have,

\[ \inf\{t > 0; \log(X_t) > 0\} = 0 < \inf\{t > 0; \log(X_t) < 0\} =: \tau. \]

Hence, for sufficiently large $N$, the stopping times 
\[ \tau_N := \tau \wedge 1/N \]

satisfy $P(\{\tau_N < \tau\}) > 0$. As 
\[ P(\{X_{\tau_N} > 1\}) = P(\{W_{1/N} \neq -(1/N)\alpha\} \cap \{\tau_N < \tau\}) = P(\{\tau_N < \tau\}) > 0 \]

and $X_{\tau_N} = 1$ on $\{\tau_N = \tau\}$, the strategy $\Phi_t = 1_{(0,\tau_N]}$ is a simple arbitrage with wealth process 
\[ V_t(\Phi; 0) = X_t \wedge \tau_N - X_0 \to \begin{cases} \exp\{W_{1/N} + (1/N)\alpha\} - 1, & \tau_N < \tau \\ 0, & \tau_N = \tau \end{cases} \quad (t \to \infty) \]

for sufficiently large $N$. Here, the arbitrage can be obtained by trading at arbitrarily short time intervals. Moreover, it is 0-admissible, because $X_t - X_0 \geq 0$ on $[0, \tau]$. 

5
These two types of arbitrages are particularly favourable for an investor: Obvious arbitrages guarantee a minimum riskless gain, if the investor starts to trade at all; 0-admissible arbitrages can be obtained without running into debt while waiting for the riskless gain.

The main result of this section shows that, if there is a simple arbitrage then there must be one of these two favourable types.

Before we state and prove the result we first introduce the notion of no obvious arbitrage on an infinite time horizon. The definition is in the spirit of Guasoni et al. [16]. Note also that ‘half of the condition’ on finite time horizons was introduced by Bayraktar and Sayit [2] in their study of simple arbitrage for non-negative, strict local martingales.

**Definition 2.3.** $X$ satisfies no obvious arbitrage (NOA), if for every stopping time $\sigma$ and for every $\epsilon > 0$ we have: If $P(\{\sigma < \infty\}) > 0$, then

$$P(\{\sigma < \infty\} \cap \{ \inf_{t \in [\sigma, \infty)} X_t > X_\sigma - \epsilon \}) > 0$$

and

$$P(\{\sigma < \infty\} \cap \{ \sup_{t \in [\sigma, \infty)} X_t < X_\sigma + \epsilon \}) > 0$$

This property means that, given a stopping time $\sigma$, the probability that the stock will not exceed (fall below) the $\epsilon$-shifted level $X_\sigma + \epsilon$ after time $\sigma$ is positive.

The next straightforward proposition explains how to obtain an obvious arbitrage, if (NOA) is violated.

**Proposition 2.4.** If $X$ does not satisfy (NOA), then $X$ has a simple arbitrage.

**Proof.** The argument is similar as the proof of necessity in Proposition 1 of Bayraktar and Sayit [2]. Without any real loss of generality we assume that condition (2) in (NOA) is violated, i.e. there is a stopping time $\sigma$ and an $\epsilon > 0$ such that $P(\{\sigma < \infty\}) > 0$ and

$$P(\{\sigma < \infty\} \cap \{ \sup_{t \in [\sigma, \infty)} X_t < X_\sigma + \epsilon \}) = 0$$

We fix a sufficiently large $K$ such that $P(\{\sigma \leq K\}) > 0$ and define the stopping time $\rho := \inf\{t \geq \sigma \wedge K; X_t \geq X_\sigma + \epsilon/2\}$, which is a.s. finite on the set $\{\sigma \leq K\}$. Then, with $\tau := \rho 1_{[\sigma \leq K]} + K 1_{[\sigma > K]}$, $1_{[\sigma \wedge K, \tau]}$ is a simple arbitrage. Note, that $V_\infty(1_{[\sigma \wedge K, \tau]}) \geq \epsilon/2$ on $\{\sigma \wedge K < \tau\}$. So this arbitrage is obvious in the terminology of Example 2.2, (i).
Theorem 2.5. Suppose $X$ has right-continuous paths. Then the following assertions are equivalent:

(i) $X$ is free of arbitrage with simple strategies.

(ii) $X$ satisfies (NOA) and $X$ has no 0-admissible arbitrage of the form $\pm 1_{(\sigma, \tau]}$ with bounded stopping times $\sigma \leq \tau$.

As a preparation we prove two propositions which are interesting in their own rights.

Proposition 2.6. Suppose $X$ has right-continuous paths. If (NOA) holds, then every simple arbitrage is 0-admissible.

Proof. Suppose $\Phi_t = \phi_0 1_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j 1_{(\tau_j, \tau_{j+1}]}$ is a simple arbitrage which is not zero admissible. We define

$$j_0 = \max \left\{ j = 0, \ldots, n - 1; \quad P \left( \inf_{t \in [\tau_j, \tau_{j+1})} V_t(\Phi; 0) < 0 \right) > 0 \right\}.$$  

Setting $\tau := \tau_{j_0 + 1}$ we observe that $V_\tau(\Phi; 0) \geq 0$ $P$-almost surely. Moreover, there is a $\delta > 0$ such that

$$P \left( \inf_{t \in (\tau_{j_0}, \tau)} V_t(\Phi; 0) \leq -2\delta \right) > 0. \quad (3)$$

Define a stopping time $\rho$ by

$$\rho = \inf \{ t > \tau_{j_0}; \quad V_t(\Phi; 0) \leq -\delta \} \wedge \tau.$$

By right-continuity of $X$ (and hence $V(\Phi; 0)$), we have $V_\rho(\Phi; 0) \leq -\delta$ on $\{\rho < \tau\}$. The latter set has positive probability by (3).

We now distinguish two cases: Let us first assume that

$$P(\{\rho < \tau\} \cap \{0 < \phi_{j_0} \leq M\}) > 0 \quad (4)$$

for some sufficiently large $M$, which is fixed from now on. Then, we have on $A := \{\rho < \tau\} \cap \{0 < \phi_{j_0} \leq M\} \in \mathcal{F}_\rho$,

$$\delta \leq V_\tau(\Phi; 0) - V_\rho(\Phi; 0) = \phi_{j_0}(X_\tau - X_\rho) \leq M(X_\tau - X_\rho).$$

Consequently,

$$P( A \cap \{ \sup_{t \in [\rho, \infty)} X_t < X_\rho + \delta/M \} ) \leq P( A \cap \{ X_\tau < X_\rho + \delta/M \} ) = 0.$$  

Defining the stopping time

$$\sigma(\omega) = \begin{cases} \rho(\omega), & \omega \in A \\ \infty, & \omega \in A^c \end{cases},$$  

7
we get

\[ P(\{ \sigma < \infty \} \cap \{ \sup_{t \in [\sigma, \infty]} X_t < X_{\sigma} + \delta/M \}) = 0 \]

in contradiction to (2) in the definition of (NOA).

If (4) does not hold, then

\[ P(\{ \rho < \tau \} \cap \{-M \leq \phi_{j_0} < 0\}) > 0 \] (5)

for some sufficiently large \( M \). Indeed, otherwise \( P(\phi_{j_0} = 0 | \rho < \tau) = 1 \), which contradicts \( V_{\rho}(\Phi; 0) < 0 \leq V_{\tau}(\Phi; 0) \) on \( \{ \rho < \tau \} \). An analogous argument as in the first case now shows that (5) is in conflict with (1) in the definition of (NOA).

**Proposition 2.7.** Suppose \( X \) is right-continuous. If \( X \) has a \( 0 \)-admissible simple arbitrage, then it has a \( 0 \)-admissible arbitrage of the form \( \pm 1_{(\sigma, \tau]} \) with bounded stopping times \( \sigma \leq \tau \).

In particular, this proposition shows that the study of \( 0 \)-admissible arbitrage can be restricted to compact time intervals.

**Proof.** Suppose \( \Phi_t = \phi_0 1_{[0]}(t) + \sum_{j=0}^{n-1} \phi_j 1_{(\tau_j, \tau_{j+1}]} \) is a \( 0 \)-admissible simple arbitrage. We define

\[ j_0 = \max \left\{ j = 0, \ldots, n - 1; P( V_{\tau_j}(\Phi; 0) = 0) = 1 \right\} . \]

We consider the strategy \( \Phi_t = \phi_{j_0} 1_{(\tau_{j_0}, \tau_{j_0+1}]} \). As \( P(V_{\tau_{j_0}}(\Phi; 0) = 0) = 1 \), we obtain

\[ V_t(\Phi; 0) = \begin{cases} 0, & t \leq \tau_{j_0} \\ V_t(\Phi; 0), & \tau_{j_0} < t \leq \tau_{j_0+1} \\ V_{\tau_{j_0+1}}(\Phi; 0), & t > \tau_{j_0+1}. \end{cases} \]

By the definition of \( i_0 \) and the fact that \( \Phi \) is \( 0 \)-admissible, we conclude that \( \Phi \) is a \( 0 \)-admissible arbitrage, and so is \( \text{sign}(\phi_{j_0}) 1_{(\tau_{j_0}, \tau_{j_0+1}]} \). Hence, one of the strategies \( 1_{(\tau_{j_0}, \tau_{j_0} + \eta]} \) or \( -1_{(\tau_{j_0}, \tau_{j_0} - \eta]} \) is a \( 0 \)-admissible arbitrage, where

\[ \tau_{\pm} = \begin{cases} \tau_{j_0 + 1}, & \pm \phi_{j_0} > 0 \\ \tau_{j_0}, & \text{otherwise}. \end{cases} \]

Thus, we have constructed a \( 0 \)-admissible arbitrage of the form \( \pm 1_{(\sigma, \tau]} \) for a.s. finite stopping times \( \sigma \leq \tau \). If \( \tau \) is bounded, the assertion is proved. Otherwise, we only treat the case that the \( 0 \)-admissible arbitrage is of the form \( 1_{(\sigma, \tau]} \). We now consider the strategies \( 1_{(\sigma \land K, \tau \land K]} \) for \( K \in \mathbb{N} \). Then,

\[ V_t(1_{(\sigma \land K, \tau \land K]}; 0) = X_{\tau \land K \land t} - X_{\sigma \land K \land t} = V_{1_{(\sigma, \tau]}; 0}. \]

Consequently, \( 1_{(\sigma \land K, \tau \land K]} \) is \( 0 \)-admissible. As

\[ \{ V_{\infty}(1_{(\sigma, \tau]}; 0) > 0 \} \cap \{ \tau \leq K \} \uparrow \{ V_{\infty}(1_{(\sigma, \tau]}; 0) > 0 \} \quad (K \uparrow \infty), \]

8
we get
\[ P(\{V_\infty(1_{(\sigma,\tau]}; 0) > 0\} \cap \{\tau \leq K\}) > 0 \]
for sufficiently large \(K\). Now, \(V_\infty(1_{(\sigma,\tau]}; 0) = V_\infty(1_{(\sigma \wedge K, \tau \wedge K]}; 0)\) on \(\{\tau \leq K\}\), which implies that
\[ P(\{V_\infty(1_{(\sigma \wedge K, \tau \wedge K]}; 0) > 0\} \cap \{\tau \leq K\}) > 0. \]
Thanks to the 0-admissibility of \(1_{(\sigma \wedge K, \tau \wedge K]}\) we conclude that this strategy is an arbitrage.

With these propositions at hand, the proof of Theorem 2.5 is immediate:

**Proof of Theorem 2.5.** (ii) \(\Rightarrow\) (i) immediately follows from Propositions 2.6 and 2.7.
(i) \(\Rightarrow\) (ii): It suffices to show that (NOA) is a necessary condition for absence of simple arbitrage, which is the assertion of Proposition 2.4.

As a corollary we obtain an infinite time horizon version of a result by Bayraktar and Sayit [2] for local martingales.

**Corollary 2.8.** Suppose \(X\) is right-continuous and there is a probability measure \(Q\) equivalent to \(P\) such that \(X\) is a \(Q\)-local martingale. Then, the following assertions are equivalent:
(i) \(X\) has no simple arbitrage.
(ii) \(X\) satisfies (NOA).

**Proof.** In view of Theorem 2.5 it suffices to show, that the existence of an equivalent local martingale measure rules out the existence of a 0-admissible arbitrage of the form \(\pm 1_{(\sigma,\tau]}\) with bounded stopping times \(\sigma \leq \tau\). This follows from a routine application of the optional sampling theorem applied to the \(Q\)-supermartingale \(V_t(\pm 1_{(\sigma,\tau]}; 0)\), which is justified by the boundedness of \(\tau\).

### 3 A characterization of simple arbitrage for continuous processes

Throughout this section we assume that the stock model \(X\) has continuous paths. Under this assumption we will characterize absence of 0-admissible simple arbitrage. In this way we will achieve a second characterization of simple arbitrage. To this end we introduce the concept of infinitesimal up’n’down.

**Definition 3.1.** Suppose \(\sigma\) is an a.s. finite stopping time and let
\[
\sigma^+ = \inf\{t \geq \sigma, \ X_t > X_\sigma\} \\
\sigma^- = \inf\{t \geq \sigma, \ X_t < X_\sigma\}.
\]
(i) $X$ satisfies infinitesimal up’n’down at $\sigma$, if
\[
\sigma^+ = \sigma^- \quad P - a.s.
\] (6)

(ii) $X$ satisfies infinitesimal up’n’down (IUD) at bounded stopping times (at a.s. finite stopping times), if it satisfies infinitesimal up’n’down at every bounded (a.s. finite) stopping time $\sigma$.

(IUD) is a condition on the fine structure of the paths. Whenever the stock price moves from the level $X_\sigma$, it will cross $X_\sigma$ infinitely often in time intervals of length $\epsilon$ for every $\epsilon > 0$.

**Proposition 3.2.** Suppose $X$ is continuous. Then, the following assertions are equivalent:
(i) $X$ satisfies (IUD) at a.s. finite stopping times.
(ii) $X$ satisfies (IUD) at bounded stopping times.
(iii) $X$ has no 0-admissible arbitrage of the form $\pm 1_{(\sigma, \tau]}$ with bounded stopping times $\sigma$ and $\tau$.
(iv) $X$ has no 0-admissible simple arbitrage.

**Proof.** We first introduce the notation
\[
\sigma_n^+ = \inf\{t \geq \sigma, \quad X_t \geq X_\sigma + 1/n\} \\
\sigma_n^- = \inf\{t \geq \sigma, \quad X_t \leq X_\sigma - 1/n\}
\] (7)
for $n \in \mathbb{N}$. Notice that $\sigma_n^\pm \downarrow \sigma^\pm$ $P$-almost surely as $n \to \infty$, because e.g. for every $\epsilon > 0$
\[
P(\{\sigma_n^+ - \sigma^+ > \epsilon\}) = P(\{\sup_{t \in [0,\epsilon]} X_{t+\sigma^+} - X_\sigma < 1/n\})
\downarrow P(\{\sup_{t \in [0,\epsilon]} X_{t+\sigma^+} - X_\sigma \leq 0\}) = 0,
\]
and the sequence $\sigma_n^+$ is non-increasing.

(i) $\Rightarrow$ (ii): obvious.

(ii) $\Rightarrow$ (iii): Suppose a strategy of the form $1_{(\sigma, \tau]}$ with a.s. finite stopping times $\sigma \leq \tau$ is an arbitrage. Of course, we can and shall assume $P(\{\tau > \sigma\}) > 0$.

We first consider the case $P(\{\sigma^- = \sigma\}|\{\tau > \sigma\}) = 1$: Then $\sigma_n \downarrow \sigma$ on $\{\tau > \sigma\}$ and thus $\tau_n := \tau \wedge \sigma_n^- \downarrow \sigma$ $P$-a.s. Hence, $P(\{\sigma < \tau_n < \tau\}) > 0$ for sufficiently large $n$. For such an $\tau_n$ we have, on $\{\sigma < \tau_n < \tau\}$
\[
V_{\tau_n}(1_{(\sigma, \tau]}|0) = X_{\tau \wedge \tau_n} - X_{\sigma \wedge \tau_n} = X_{\sigma_n^-} - X_\sigma = -1/n.
\]
Thus, $1_{(\sigma, \tau]}$ is not 0-admissible. Note that in the first case we did not assume boundedness of $\sigma$ and $\tau$, and did not not apply (IUD).
Now suppose that $P\{\sigma^- = \sigma\}|\{\tau > \sigma\} < 1$ and $\sigma$ is bounded. We observe that, thanks to (IUD) at the bounded stopping time $\sigma$ and the continuous paths of $X$, $X_t = X_{\sigma}$ on $[\sigma, \sigma^-]$ and, hence, $V_t(1_{(\sigma, \tau]}(0)) = X_{\tau \wedge t} - X_{\sigma \wedge t} = 0$ for $t \in [0, \sigma^-]$. If $1_{(\sigma, \tau]}$ is a $0$-admissible arbitrage then so is $1_{(\sigma^- \wedge \tau, \tau]}$. However $(\sigma^-)^- = \sigma^-$, and so the first case applies.

The argument for arbitrages of the form $1_{(\sigma, \tau]}$ is essentially the same.

(iii) $\Rightarrow$ (iv): Proposition 2.7.

(iv) $\Rightarrow$ (i): Suppose that $X$ does not satisfy (IUD) at some a.s. finite stopping time $\sigma$. Without any real loss of generality we assume that the set $A = \{\omega, \sigma^+(\omega) < \sigma^-(\omega)\}$ has strictly positive probability. Note that $A \in F_{\sigma^+}$. We define the sequence of stopping times

$$\tau_n = (\sigma^- \wedge \sigma_n^+) 1_A + \sigma^+ 1_{A^c}.$$ 

Then, $\tau_n \geq \sigma^+$ a.s. and $\tau_n > \sigma^+$ on $A$. By construction and continuity of $X$, we have $X_t \geq X_{\sigma}$ for $t \in [\sigma^+, \tau_n]$. Therefore the strategies $1_{(\sigma^+, \tau_n]}$, $n \in \mathbb{N}$, are $0$-admissible. As $\sigma_n^+ \downarrow \sigma^+ \text{P-a.s.}$, we get $\tau_n \downarrow \sigma^+ \text{P-a.s.}$ Therefore,

$$P\{\sigma^+ < \tau_n < \sigma^-\} = P(A \cap \{\tau_n < \sigma^-\}) > 0$$

for sufficiently large $n$. However, on $\{\sigma^+ < \tau_n < \sigma^-\}$,

$$V_{\infty}(1_{(\sigma^+, \tau_n]}(0)) = X_{\tau_n} - X_{\sigma^+} = X_{\sigma_n^+} - X_{\sigma} = 1/n.$$

Consequently, $1_{(\sigma^+, \tau_n]}$ is a $0$-admissible arbitrage for sufficiently large $n$. $\square$

A combination of the previous proposition with Theorem 2.5 yields the following characterization of simple arbitrage for continuous stock models.

**Theorem 3.3.** Suppose $X$ is continuous. Then, the following assertions are equivalent:

(i) $X$ does not admit a simple arbitrage.

(ii) $X$ satisfies (IUD) at bounded stopping times and (NOA).

4 A closer look at infinitesimal up’n’down

In this section we investigate the infinitesimal up’n’down property in some more detail. As a main result we will derive the following sufficient condition.

**Theorem 4.1.** Suppose $X_t = M_t + Y_t$, where $M$ is a continuous $(\mathcal{F}_t)$-local martingale and $Y_t$ is an $(\mathcal{F}_t)$-adapted process which is locally $1/2$-Hölder continuous with respect to the quadratic variation $\langle M \rangle$ of $M$, in one of the following senses: For every $K > 0$ there is a non-negative random variable $C_K$ with

$$P\left(\left\{\forall 0 \leq t \leq s, (M)_t \leq K | Y_s - Y_t | \leq C_K | \langle M \rangle_s - \langle M \rangle_t |^{1/2}\right\}\right) = 1$$
or
\[
P \left( \left\{ \forall 0 \leq t \leq s \leq K \mid \left| Y_s - Y_t \right| \leq C_K \left| \langle M \rangle_s - \langle M \rangle_t \right|^{1/2} \right\} \right) = 1
\]

Then, \( X \) satisfies (IUD) (with respect to \( \mathcal{F}_t \)).

In the above statement and in the rest of the paper, the phrase ‘\( X \) satisfies (IUD)’ refers to each of the two equivalent notions of (IUD) in Proposition 3.2.

**Remark 4.2.** Note that the Hölder exponent 1/2 in the above theorem is sharp. Indeed, for every \( \alpha < 1/2 \), the process \( W_t + t^\alpha \) violates (IUD) at \( \sigma = 0 \) by the law of the iterated logarithm, cp. Example 2.2 (ii) above.

Before we prove Theorem 4.1, we illustrate the two different Hölder conditions in the next example.

**Example 4.3.** Suppose the stochastic process \( Y_t \) is 1/2-Hölder continuous on compacts in the usual sense, i.e. for every \( K > 0 \) there is a non-negative random variable \( C_K \) such that
\[
P \left( \left\{ \forall 0 \leq t \leq s \leq K \mid \left| Y_s - Y_t \right| \leq C_K \left| s - t \right|^{1/2} \right\} \right) = 1.
\]

Many well-studied processes such as a fractional Brownian motion with Hurst parameter \( H > 1/2 \) satisfy 1/2-Hölder continuity in this sense. Then,

(i) We run \( Y \) in the stochastic clock \( \langle M \rangle_t \) given by the quadratic variation of some continuous \( (\mathcal{F}_t) \) \( t \in [0, \infty) \)-local martingale \( M \). Then \( Y_{\langle M \rangle_t} \) satisfies the first Hölder condition of Theorem 4.1, i.e. for every \( K > 0 \)
\[
P \left( \left\{ \forall 0 \leq t \leq s, \langle M \rangle_s \leq K \mid Y_{\langle M \rangle_s} - Y_{\langle M \rangle_t} \leq C_K \left| \langle M \rangle_s - \langle M \rangle_t \right|^{1/2} \right\} \right) = 1.
\]

Consequently, \( M_t + Y_{\langle M \rangle_t} \) satisfies (IUD) with respect to \( (\mathcal{F}_t)_{t \in [0, \infty)} \), if \( Y_{\langle M \rangle} \) is \( (\mathcal{F}_t)_{t \in [0, \infty)} \)-adapted. So the first Hölder condition is particularly suitable in a time change setting.

(ii) If \( Y \) is \( (\mathcal{F}_t)_{t \in [0, \infty)} \)-adapted and \( M \) is a continuous \( (\mathcal{F}_t)_{t \in [0, \infty)} \)-local martingale with
\[
\langle M \rangle_s - \langle M \rangle_t \geq \int_t^s \sigma_u^2 \, du, \quad 0 \leq t \leq s,
\]
for an adapted process \( \sigma \) with \( \inf_{0 \leq u \leq K} |\sigma_u| > 0 \) \( P \)-almost surely for every \( K > 0 \). Then, \( P \)-almost surely for every \( 0 \leq t \leq s \leq K \),
\[
\left| Y_s - Y_t \right| \leq C_K |t - s|^{1/2} \leq C_K (\inf_{0 \leq u \leq K} |\sigma_u|)^{-1} \left( \int_t^s \sigma_u^2 \, du \right)^{1/2} \leq C_K (\inf_{0 \leq u \leq K} |\sigma_u|)^{-1} |\langle M \rangle_s - \langle M \rangle_t|^{1/2}.
\]

Hence, \( Y \) fulfills the second Hölder condition in Theorem 4.1 and, thus, \( M_t + Y_t \) satisfies (IUD) with respect to \( (\mathcal{F}_t)_{t \in [0, \infty)} \). So, the second Hölder
condition is particularly suitable for perturbing stochastic volatility models driven by $M_t = \int_0^t \sigma_u dW_u$ with a Brownian motion $W$ and a strictly positive volatility process $\sigma$ by an $1/2$-Hölder continuous process. This example will be discussed in more detail in Section 6 below.

We first give the proof of (a slightly more general assertion than) Theorem 4.1 in the case that the local martingale is a Brownian motion. The corresponding result is singled out in the following proposition.

**Proposition 4.4.** Suppose $X_t = W_t + Z_t$, where $W$ is an $(\mathcal{F}_t)$-Brownian motion and $Z_t$ is an $(\mathcal{F}_t)$-adapted processes, such that there are two non-decreasing sequences $K_n$ and $C_n$ of non-negative random variables with $K_n \uparrow \infty$ $P$-almost surely and

\[
P\left(\bigcap_{0 \leq s \leq t \leq K_n} |Z_s - Z_t| \leq C_n|s - t|^{1/2}\right) = 1
\]

for every $n \in \mathbb{N}$. Then, $X$ satisfies (IUD) (with respect to $(\mathcal{F}_t)$).

**Proof.** Suppose $\sigma$ is an a.s. finite stopping time. By assumption, there is a set $\Omega'$ of full $P$-measure such that for every $n \in \mathbb{N}$ and every $\omega \in \Omega'$

\[
\sup_{0 \leq s \leq t \leq K_n(\omega)} |Z_s(\omega) - Z_t(\omega)| \leq C_n(\omega)|s - t|^{1/2}. \tag{8}
\]

In particular, $P$-almost every path of $Z$ is continuous. Moreover, $W^\sigma_t = W_{\sigma+t} - W_\sigma$ is a Brownian motion, see e.g. Theorem 2.6.16 in Karatzas and Shreve [19]. By (8) and the law of the iterated logarithm (see e.g. Theorem 2.9.23 in [19]) applied to $W^\sigma$ there is a set $\Omega'' \subset \Omega'$ of full $P$-measure such that for every $n \in \mathbb{N}$ and $\omega \in \Omega'' \cap \{\sigma + 1 \leq K_n\}$

\[
\limsup_{t \downarrow 0} \frac{X_{\sigma+t}(\omega) - X_\sigma(\omega)}{\sqrt{2t \log \log(1/t)}} = \limsup_{t \downarrow 0} \left( \frac{W^\sigma_t(\omega)}{\sqrt{2t \log \log(1/t)}} + \frac{Z_{\sigma+t}(\omega) - Z_\sigma(\omega)}{\sqrt{2t \log \log(1/t)}} \right) = \limsup_{t \downarrow 0} \frac{W^\sigma_t(\omega)}{\sqrt{2t \log \log(1/t)}} = 1
\]

and, analogously, for $\omega \in \Omega'' \cap \{\sigma + 1 \leq K_n\}$

\[
\liminf_{t \downarrow 0} \frac{X_{\sigma+t}(\omega) - X_\sigma(\omega)}{\sqrt{2t \log \log(1/t)}} = -1.
\]

Consequently, we obtain, for every $n \in \mathbb{N}$ and $\omega \in \Omega'' \cap \{\sigma + 1 \leq K_n\}$,

\[
\sigma^+(\omega) = \sigma(\omega) = \sigma^-(\omega).
\]

As $\bigcup_n \{\sigma + 1 \leq K_n\}$ has full $P$-measure, the proof is complete. \qed
The proof of Theorem 4.1 in the general case can be obtained by a time change argument which we prepare by the following lemma.

**Lemma 4.5.** Suppose that $X_t$ and $A_t$ are $(\mathcal{F}_t)$-adapted continuous processes, $A$ is non-decreasing with $A_0 = 0$ and $\lim_{t \to \infty} A_t = \infty$. Define the $(\mathcal{F}_t)$-stopping times

$$T(s) = \inf\{t \geq 0; \ A_t > s\}; \quad s \geq 0,$$

and suppose that $X_{T(A_t)} = X_t$ for every $t \geq 0$.

If $X_{T(t)}$ is a continuous process and $(X_{T(t)}, \mathcal{F}_{T(t)})$ satisfies (IUD), then so does $(X_t, \mathcal{F}_t)$.

**Proof.** Note that $\{T(t) < s\} = \{A_s > t\} \in \mathcal{F}_s$ and, hence, the random times $T(t)$ are $(\mathcal{F}_t)_{t \in [0, \infty)}$-stopping times indeed. Suppose that $\sigma$ is an a.s. finite $(\mathcal{F}_t)_{t \in [0, \infty)}$-stopping time. Then,

$$\{A_\sigma > t\} = \{\sigma > T(t)\} \in \mathcal{F}_{T(t)},$$

i.e. $A_\sigma$ is an a.s. finite $(\mathcal{F}_{T(t)})_{t \in [0, \infty)}$-stopping time. We can now define the $(\mathcal{F}_{T(t)})_{t \in [0, \infty)}$-stopping time

$$A^+_\sigma := \inf\{t > A_\sigma; \ X_{T(t)} > X_{T(A_\sigma)}\} = \inf\{t > A_\sigma; \ X_{T(t)} > X_\sigma\}.$$

We shall prove that

$$\sigma^+ = T(A^+_\sigma),$$

where the $(\mathcal{F}_t)_{t \in [0, \infty)}$-stopping time $\sigma^+$ was defined in Definition 3.1.

In order to prove the inequality $\sigma^+ \geq T(A^+_\sigma)$, it suffices to show that

$$X_t \leq X_\sigma \quad \text{for} \quad t \in [\sigma, T(A^+_\sigma)].$$

Since

$$T(A_\sigma) = \sup\{u \geq \sigma; \ A_u = A_\sigma\} = \inf\{u \geq \sigma; \ A_u > A_\sigma\},$$

we get $A_t = A_\sigma$ for $t \in [\sigma, T(A_\sigma)]$ and, thus,

$$X_t = X_{T(A_t)} = X_{T(A_\sigma)} = X_\sigma \quad \text{for} \quad t \in [\sigma, T(A_\sigma)].$$

Now suppose that $t \in [T(A_\sigma), T(A^+_\sigma)]$. Then, $A_t \in [A_\sigma, A^+_\sigma]$, because $A_{T(s)} = s$. By the definition of $A^+_\sigma$, we thus obtain

$$X_t = X_{T(A_t)} \leq X_\sigma \quad \text{for} \quad t \in [T(A_\sigma), T(A^+_\sigma)].$$

This completes the proof of the inequality $\sigma^+ \geq T(A^+_\sigma)$.

For the reverse inequality, we define

$$(A^+_\sigma)_n := \inf\{t > A_\sigma; \ X_{T(t)} > X_{T(A_\sigma)} + 1/n\} = \inf\{t > A_\sigma; \ X_{T(t)} > X_\sigma + 1/n\}$$
and recall the definition of $\sigma^+_n$ in (7). By the continuity of $X_T(t)$, we get

$$X_T((A^+_n)_n) = X_\sigma + 1/n \quad \text{on} \quad \{(A^+_n)_n < \infty\}$$

Thus, $\sigma^+_n \leq T((A^+_n)_n)$. As $(A^+_n)_n \to A_\sigma^+$ for $n \to \infty$ on $\{A_\sigma^+ < \infty\}$ and $T$ is rightcontinuous, we get

$$\sigma^+ = \lim_{n \to \infty} \sigma^+_n \leq \lim_{n \to \infty} T((A^+_n)_n) = T(A_\sigma^+) \quad \text{on} \quad \{A_\sigma^+ < \infty\}.$$ 

As the inequality $\sigma^+ \leq T(A_\sigma^+)$ is trivial on $\{A_\sigma^+ = \infty\}$, the proof of $\sigma^+ = T(A_\sigma^+)$ is complete.

Analogously, one can show

$$\sigma^- = T(A^-_\sigma)$$

Now we apply (IUD) for the process $(X_T(t), F_T(t))$ at the stopping time $A_\sigma$ and obtain

$$A^+_\sigma = A^-_\sigma.$$

Consequently,

$$\sigma^+ = T(A^+_\sigma) = T(A^-_\sigma) = \sigma^-,$$

i.e. (IUD) holds at $\sigma$ for the process $(X_t, F_t)$.

We will also apply the following lemma in the proof of Theorem 4.1

**Lemma 4.6.** The continuous process $(X_t, F_t)$ satisfies (IUD), if and only if $(X_{t\wedge T}, F_{t\wedge T})$ satisfies (IUD) for every $T > 0$.

**Proof.** Fix $T > 0$. Then, $\tau$ is an $(F_t)_{t \in [0, \infty)}$-stopping time bounded by $T$, if and only if it is an $(F_{t\wedge T})_{t \in [0, \infty)}$-stopping time bounded by $T$. For such a stopping time, it obviously holds that $X_\tau = X_{\tau \wedge T}$. Therefore, $(X_t, F_t)$ has a 0-admissible arbitrage of the form $\pm 1_{[\sigma, \tau]}$ with $(F_t)_{t \in [0, \infty)}$-stopping times $\sigma \leq \tau \leq T$, if and only if $(X_{t\wedge T}, F_{t\wedge T})$ has a 0-admissible arbitrage of the form $\pm 1_{[\sigma, \tau]}$ with $(F_{t\wedge T})_{t \in [0, \infty)}$-stopping times $\sigma \leq \tau \leq T$. The latter statement is equivalent to the assertion that $(X_{t\wedge T}, F_{t\wedge T})$ has a 0-admissible arbitrage of the form $\pm 1_{[\sigma, \tau]}$ with $(F_{t\wedge T})_{t \in [0, \infty)}$-stopping times $\sigma \leq \tau$. Consequently, $(X_t, F_t)$ has no 0-admissible arbitrage of the form $\pm 1_{[\sigma, \tau]}$ with bounded stopping times, if and only if for every $T > 0$, $(X_{t\wedge T}, F_{t\wedge T})$ has no 0-admissible arbitrage of the form $\pm 1_{[\sigma, \tau]}$ with a.s. finite stopping times. Therefore the claim of the lemma follows from Proposition 3.2. \qed

We can now give the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Case 1: $\langle M \rangle_\infty = \infty$ $P$-almost surely. Define

$$T(s) := \inf\{t \geq 0; \langle M \rangle_t > s\}.$$
By the Dambis-Dubins-Schwarz Theorem (see Karatzas and Shreve [19, Theorem 3.4.6]), the process $W_t = M_{T(t)}$ is an $(\mathcal{F}_{T(t)})_{t \in [0, \infty)}$-Brownian motion and $W_{\langle M \rangle_t} = M_t$. We define $Z_t = Y_{T(t)}$. Each of the Hölder conditions of $Y$ in terms of the quadratic variation of $M$ makes sure that

$$P(\{\forall s < t \quad \langle M \rangle_t = \langle M \rangle_s \Rightarrow Y_t = Y_s\}) = 1.$$ 

As $T(\langle M \rangle_t) = \sup\{u \geq t \mid \langle M \rangle_u = \langle M \rangle_t\}$, we conclude that $Z(\langle M \rangle_t) = Y_{T(\langle M \rangle_t)} = Y_t$, $P$-almost surely. Consequently,

$$X_{T(t)} = W_t + Z_t$$

satisfies $X_{T(\langle M \rangle_t)} = X_t$. In order to apply Lemma 4.5 it remains to show continuity of $Z_t = Y_{T(t)}$. If the first Hölder condition is satisfied, then we go on as follows. For $t \leq s \leq K$ with fixed $K > 0$, we have $t = \langle M \rangle_{T(t)} \leq \langle M \rangle_{T(s)} = s \leq K$. Hence, by the Hölder condition of $Y$ in terms of the quadratic variation of $M$ we obtain

$$P\left(\left\{\forall 0 \leq t \leq s \leq K \mid Y_{T(t)} - Y_{T(s)} \leq C_K|t - s|^{1/2}\right\}\right) = 1.$$ 

Thus, $Z$ is continuous and satisfies the Hölder condition in Proposition 4.4. By this proposition, $X_{T(t)} = W_t + Z_t$ then satisfies (IUD) with respect to the filtration $(\mathcal{F}_{T(t)})$. Lemma 4.5 now yields (IUD) for $(X_t, \mathcal{F}_t)$.

If the second Hölder condition is fulfilled by $Y$, the argumentation can be modified as follows. Suppose $0 \leq t \leq s \leq \langle M \rangle_K$ for some fixed $K > 0$. Then, $T(t) \leq T(s) \leq T(\langle M \rangle_K)$. As $Y_u = Y_K$ for $u \in [K, T(\langle M \rangle_K)]$, we get, $P$-almost surely for every $0 \leq t \leq s \leq \langle M \rangle_K$,

$$|Y_{T(t)} - Y_{T(s)}| = |Y_{T(t) \wedge K} - Y_{T(s) \wedge K}| \leq C_K|\langle M \rangle_{T(t) \wedge K} - \langle M \rangle_{T(s) \wedge K}|^{1/2},$$

making use of the second Hölder condition. Since $\langle M \rangle_u = \langle M \rangle_K$ for $u \in [K, T(\langle M \rangle_K)]$, and $\langle M \rangle_{T(v)} = v$ for every $v \geq 0$, we finally obtain, for every $K > 0$

$$P\left(\left\{\forall 0 \leq t \leq s \leq \langle M \rangle_K \mid Y_{T(t)} - Y_{T(s)} \leq C_K|t - s|^{1/2}\right\}\right) = 1.$$ 

As $\langle M \rangle_K \uparrow \infty$ for $K \to \infty$, the assumptions of Proposition 4.4 are again satisfied and the rest of the proof is identical to the case of the first Hölder condition.

**Case 2:** $\langle M \rangle_\infty < \infty$ with positive probability: If necessary we enlarge the probability by adding a Brownian motion $B$ independent of $\mathcal{F}_\infty$. We consider the processes $(T > 0)$,

$$X^T_t = \begin{cases} X_t, & 0 \leq t \leq T \\ X_T + B_{t-T}, & T < t < \infty \end{cases}$$

16
in the augmented filtration \( (\mathcal{F}_t^T) \) of
\[
\mathcal{G}_t^T = \begin{cases} 
\mathcal{F}_t, & 0 \leq t \leq T \\
\mathcal{F}_t \vee \sigma(B_s, 0 \leq s \leq t - T), & T < t < \infty.
\end{cases}
\]

Note that
\[
X_t^T = M_t^T + Y_t^T
\]
with
\[
M_t^T = \begin{cases} 
M_t, & 0 \leq t \leq T \\
M_t + B_{t-T}, & T < t < \infty
\end{cases}
\quad \text{and} \quad Y_t^T = Y_{t\wedge T}.
\]

Here \( M_t^T \) is a local martingale with quadratic variation tending to infinity and \( Y_t^T \) inherits the respective Hölder condition from \( Y_t \). Therefore, by the first case, the processes \( (X_t^T, \mathcal{F}_t^T) \) satisfy (IUD). By Lemma 4.6 so do the processes \( (X_t \wedge T, \mathcal{F}_t \wedge T) \). Hence, for every \( T > 0 \), the processes \( (X_t \wedge T, \mathcal{F}_t \wedge T) \) satisfy (IUD) and the assertion follows thanks to Lemma 4.6.

We close this section by showing that the (IUD) property is stable with respect to strictly monotone transformations, shrinkage of filtration and equivalent change of measure.

**Proposition 4.7.** Suppose \( (X_t, \mathcal{F}_t) \) satisfies (IUD) under the measure \( P \). Then,
(i) (Monotone transformation) For any strictly increasing (or strictly decreasing) continuous function \( f \), the process \( (f(X_t), \mathcal{F}_t) \) satisfies (IUD).
(ii) (Shrinkage of filtration) Suppose \( \mathcal{G}_t \subset \mathcal{F}_t \) is a sub-filtration (satisfying the usual assumptions) to which \( X \) is adapted. Then, \( (X_t, \mathcal{G}_t) \) satisfies (IUD).
(iii) (Equivalent change of measure) If \( Q \) is a probability measure equivalent to \( P \), then \( (X_t, \mathcal{F}_t) \) satisfies (IUD) under the measure \( Q \).

**Proof.** (i) If \( f \) is strictly increasing we have,
\[
\sigma_+^X := \inf \{ t \geq \sigma, \ X_t > X_\sigma \} = \inf \{ t \geq \sigma, \ f(X_t) > f(X_\sigma) \} = \sigma_{f(X)}^+(X)
\]
and \( \sigma_-^X = \sigma_{f(X)}^-(X) \). Analogously, it holds that \( \sigma^+_X = \sigma^+_{f(X)} \), if \( f \) is strictly decreasing. The claim is then obvious by the definition of (IUD).
(ii) is clear, because any a.s. finite \( (\mathcal{G}_t) \)-stopping time is an \( (\mathcal{F}_t) \)-stopping time and (IUD) holds for every a.s. finite \( (\mathcal{F}_t) \)-stopping time by assumption.
(iii) is obvious by the definition.

**Remark 4.8.** In part (i) of the above proposition strict monotonicity cannot be replaced by (non-strict) monotonicity. E.g. for a Brownian motion \( W_t \) and \( f = \max\{0, x\} \), the process \( f(W_t) \) obviously does not satisfy (IUD) at \( \sigma = 0 \). In contrast, we will see in Proposition 5.6 below that the no obvious arbitrage property is preserved under composition with a (non-strictly) monotonous function \( f \).
5 A closer look at no obvious arbitrage

In this section we give some sufficient conditions for the no obvious arbitrage property (NOA) to hold. Fortunately, the well-studied concept of conditional full support for continuous processes [15, 5, 13, 21] implies (NOA) on a bounded time horizon \([0, T]\). So it is easy to provide plenty of examples with this property.

**Definition 5.1.** For every \(T > 0\), we say that \((X_t, \mathcal{F}_t)\) satisfies *no obvious arbitrage on* \([0, T]\) (NOA\(_T\)), if for every \([0, T]\)-valued stopping time \(\sigma\) and for every \(\epsilon > 0\) we have: If \(P(\{\sigma < T\}) > 0\), then

\[
P(\{\sigma < T\} \cap \{ \inf_{t \in [\sigma, T]} X_t > X_\sigma - \epsilon \}) > 0 \tag{9}
\]

and

\[
P(\{\sigma < T\} \cap \{ \sup_{t \in [\sigma, T]} X_t < X_\sigma + \epsilon \}) > 0 \tag{10}
\]

No obvious arbitrage on \([0, T]\) can easily be reformulated in terms of (NOA) as introduced in Definition 2.3 and applied in the characterization of absence of simple arbitrage in Theorems 2.5 and 3.3.

**Lemma 5.2.** Suppose \(X\) is right-continuous. Then, \((X_t, \mathcal{F}_t)\) satisfies no obvious arbitrage on \([0, T]\), if and only if \((X_{\sigma \wedge T}, \mathcal{F}_{\sigma \wedge T})\) satisfies (NOA) in the sense of Definition 2.3.

**Proof.** Suppose (9) holds and \(\sigma\) is an \((\mathcal{F}_t)_{t \geq 0}\)-stopping time with \(P(\{\sigma < \infty\}) > 0\). We decompose

\[\{\sigma < \infty\} = \{\sigma < T\} \cup \{T \leq \sigma < \infty\}\]

If \(P(\{T \leq \sigma < \infty\}) > 0\), then, for every \(\epsilon > 0\),

\[
P(\{\sigma < \infty\} \cap \{ \inf_{t \in [\sigma, \infty]} X_{\sigma \wedge T} > X_\sigma - \epsilon \}) \\
\geq P(\{T \leq \sigma < \infty\} \cap \{ \inf_{t \in [\sigma, \infty]} X_{\sigma \wedge T} > X_\sigma - \epsilon \}) \\
= P(\{T \leq \sigma < \infty\}) > 0.
\]

If \(P(\{\sigma < T\}) > 0\), then \(P(\{\sigma \wedge T < T\}) > 0\) and so we can apply (9) to the \((\mathcal{F}_t)_{0 \leq t \leq T}\)-stopping time \(\sigma \wedge T\) and obtain,

\[
P(\{\sigma < \infty\} \cap \{ \inf_{t \in [\sigma, \infty]} X_{\sigma \wedge T} > X_{\sigma \wedge T} - \epsilon \}) \\
\geq P(\{\sigma \wedge T < T\} \cap \{ \inf_{t \in [\sigma \wedge T, T]} X_t > X_{\sigma \wedge T} - \epsilon \}) > 0.
\]

Hence, (9) implies (1) for \((X_{\sigma \wedge T}, \mathcal{F}_{\sigma \wedge T})\). One can show analogously that (10) implies (2) for \((X_{\sigma \wedge T}, \mathcal{F}_{\sigma \wedge T})\).
It remains to show that (9) – (10) is necessary for \((X_t^T, \mathcal{F}_{t^T})\) having (NOA). To this end we assume that (1) holds for \((X_t^T, \mathcal{F}_{t^T})\). Given an \([0, T]\)-valued \((\mathcal{F}_t)_{t \geq 0}\)-stopping time \(\sigma\), we define the \((\mathcal{F}_{t^T})_{t \geq 0}\)-stopping time
\[
\tilde{\sigma} = \begin{cases} 
\sigma, & \sigma < T \\
\infty, & \sigma = T.
\end{cases}
\]
If \(P(\{\sigma < T\}) > 0\) we can apply (1) to \(\tilde{\sigma}\) and get
\[
P(\{\sigma < T\} \cap \{ \inf_{t \in [\sigma, T]} X_t > X_\sigma - \epsilon \}) = P(\{\tilde{\sigma} < \infty\} \cap \{ \inf_{t \in [\tilde{\sigma}, \infty)} X_{t^T} > X_{\tilde{\sigma}^T} - \epsilon \}) > 0,
\]
hence (9). Similarly, (2) for \((X_{t^T}, \mathcal{F}_{t^T})\) implies (10).

We now recall the notion of conditional full support, which was introduced by Guasoni et al. [15] in their study of trading under transaction cost.

**Definition 5.3.** An \((\mathcal{F}_t)\)-adapted continuous process \(X\) is said to have conditional full support with respect to \((\mathcal{F}_t)\) on \([0, T]\), if, for every \(t \in [0, T]\),
\[
supp(\text{Law}[X_{t \leq u \leq T}|\mathcal{F}_t]) = \mathcal{C}_{X_t}([t, T]), \quad P\text{-a.s.},
\]
where \(\mathcal{C}_x([t, T])\) denotes the space of continuous functions \(f\) on \([t, T]\) with values in \(\mathbb{R}\) and \(f(t) = x\).

By Lemma 2.9 in Guasoni et al. [15] the conditional full support property transfers from deterministic time points \(t\) to \([0, T]\)-valued stopping times \(\sigma\). Consequently we have:

**Proposition 5.4.** If a continuous process \(X\) has conditional full support with respect to \((\mathcal{F}_t)\) on \([0, T]\), then no obvious arbitrage holds for \((X_t, \mathcal{F}_t)\) on \([0, T]\).

The following remark collects some results on conditional full support:

**Remark 5.5.** (i) Brownian motion has conditional full support on \([0, T]\) for every \(T > 0\).
(ii) Conditional full support is preserved under augmentation of the filtration, see Pakkanen [21], Lemma 2.10.
(iii) If \(X\) had conditional full support with respect to its own filtration \((\mathcal{F}_t^X)\) and the continuous process \(Y\) is independent of \(X\), then \(X + Y\) has conditional full support with respect to \((\mathcal{F}_t^X \vee \mathcal{F}_t^Y)\), see Remark 6.1 of [3] for a sketch of the proof.
(iv) The conditional full support property for Gaussian processes with stationary increments is discussed by Cherny [5] and Gasbarra et al. [13]. Some interesting results of the conditional full support of stochastic integrals can
be found in Pakkanen [21].

(v) In Guasoni [14] the notion of stickiness of a stochastic process was introduced. It is straightforward to verify that conditional full support on $[0, T]$ implies stickiness on $[0, T]$, which in turn guarantees no obvious arbitrage on $[0, T]$.

We finally discuss that the no obvious arbitrage property is preserved under some standard operations.

**Proposition 5.6.** Suppose $(X_t, \mathcal{F}_t)$ is right-continuous and satisfies (NOA) under the measure $P$. Then,

(i) (Monotone transformation) For any non-decreasing (or non-increasing) continuous function $f$, the process $(f(X_t), \mathcal{F}_t)$ satisfies (NOA).

(ii) (Shrinkage of filtration) Suppose $\mathcal{G}_t \subset \mathcal{F}_t$ is a sub-filtration (satisfying the usual assumptions) to which $X$ is adapted. Then, $(X_t, \mathcal{G}_t)$ satisfies (NOA).

(iii) (Equivalent change of measure) If $Q$ is a probability measure equivalent to $P$, then $(X_t, \mathcal{F}_t)$ satisfies (NOA) under the measure $Q$.

**Remark 5.7.** In view of Lemma 5.2 (NOA) can be replaced by (NOA$_T$) in the above proposition.

**Proof.** Only the first assertion is non-trivial. Suppose $f$ is non-decreasing and $f(X)$ does not satisfy (NOA). Without any real loss of generality, we assume that (1) is violated for $f(X)$, i.e. there is an $\epsilon > 0$ and a stopping time $\sigma$ with $P(\{\sigma < \infty\}) > 0$ such that

$$P(\{\sigma < \infty\} \cap \{\inf_{t \in [\sigma, \infty)} f(X_t) > f(X_\sigma) - \epsilon\}) = 0. \quad (11)$$

We now define the stopping times

$$\sigma_K := \begin{cases} \sigma, & |X_\sigma| \leq K \\ \infty, & \text{otherwise} \end{cases}, \quad K \in \mathbb{N}.$$

Then, $P(\{\sigma_K < \infty\}) > 0$ for a sufficiently large $K$ which is fixed from now on. Obviously, (11) holds with $\sigma$ replaced by $\sigma_K$. Defining the stopping time

$$\tau = \inf\{t \geq \sigma_K; f(X_t) \leq f(X_\sigma) - \epsilon/2\},$$

we observe that $\tau < \infty$ on $\{\sigma_K < \infty\}$ and, by right-continuity of $X$,

$$f(X_\tau) \leq f(X_{\sigma_K}) - \epsilon/2 \quad \text{on} \quad \{\sigma_K < \infty\}. \quad (12)$$

By continuity of $f$, there is a $0 < \delta < 1$ such that $|f(x) - f(y)| < \epsilon/2$ for $x, y$ with $|x - y| < \delta$ and $|x|, |y| \leq K + 1$. In particular, (12) implies that

$$|X_{\sigma_K} - X_\tau| \geq \delta \quad \text{on} \quad \{\sigma_K < \infty\}. \quad 20$$
As $f$ is non-decreasing we conclude, thanks to (12), that
\[ X_{\tau} \leq X_{\sigma_K} - \delta \quad \text{on} \quad \{\sigma_K < \infty\}. \]

In particular,
\[ P(\{\sigma_K < \infty\} \cap \{\inf_{t \in [\sigma_K, \infty)} X_t > X_{\sigma_K} - \delta\}) = 0, \]
i.e. $X$ does not satisfy (NOA).

\section{Examples}

We finally present some examples of models which are free of simple arbitrage, although they may fail to be semimartingales. The models, which we discuss here, can be considered as mixed models in the sense that some well-known arbitrage-free semimartingale models are combined with some Hölder continuous processes such as fractional Brownian motion.

Throughout the section we shall work on finite time horizons. To simplify the terminology we say that a model is free of simple arbitrage with bounded stopping times, if no simple strategy with bounded stopping times is an arbitrage (instead of considering the larger class of strategies with a.s. finite stopping times). Combining Theorem 3.3 with Lemmas 4.6 and 5.2, we have:

\textbf{Corollary 6.1.} Suppose $X$ is continuous. Then the following assertions are equivalent:

(i) $X$ is free of simple arbitrage with bounded stopping times.

(ii) $X$ satisfies (IUD) and (NOA) for every $T > 0$.

\subsection{Mixed Brownian models}

Our first class of examples concerns ‘mixed Brownian models’, i.e. a Brownian motion perturbed by adding some suitable processes.

\textbf{Corollary 6.2.} Suppose $(W_t, \mathcal{F}_t)$ is a Brownian motion and $Y_t$ is an $(\mathcal{F}_t)$-adapted process independent of $W$, which is $1/2$-Hölder continuous on compacts in the sense of Example 4.3. Then, $X_t = W_t + Y_t$ is free of simple arbitrage with bounded stopping times with respect to the augmentation of the filtration $(\mathcal{F}_t^X)$ generated by $X$.

\textit{Proof.} By Proposition 4.4, $(X_t, \mathcal{F}_t)$ satisfies (IUD) and, hence, by Proposition 4.7, (IUD) also holds with respect to the augmentation of $(\mathcal{F}_t^X)$. Moreover, $X_t$ has conditional full support with respect to the augmentation of $(\mathcal{F}_t^W \lor \mathcal{F}_t^Y)$ on compact intervals by Remark 5.5. Then, by Propositions 5.4 and 5.6, $X_t$ satisfies (NOA) for every $T > 0$ with respect to the augmentation of $(\mathcal{F}_t^X)$. Now Corollary 6.1 concludes. \hfill $\Box$
Example 6.3. We first provide an example which illustrates that the Hölder exponent in Corollary 6.2 is sharp. Suppose \( X_t = W_t + t^\alpha \) for a Brownian motion \( W \) and an \( \alpha > 0 \). Then, \( X \) is free of simple arbitrage with bounded stopping times, if and only if \( \alpha \geq 1/2 \). Indeed, if \( \alpha < 1/2 \), (IUD) is violated at \( \sigma = 0 \) and hence a 0-admissible simple arbitrage can be constructed as shown in Example 2.2. For \( \alpha \geq 1/2 \) the previous corollary applies. Note that \( X \) has an equivalent martingale measure for \( \alpha > 1/2 \) (and so absence of arbitrage is classical for this range of parameters). Different proofs for absence of simple arbitrage with bounded stopping times for \( \alpha = 1/2 \) can also be found in Delbaen and Schachermayer [12] and Jarrow et al. [18]. Delbaen and Schachermayer [12] also construct an nds-admissible arbitrage for \( W_t + t^{1/2} \) in the larger class of strategies with continuous readjustment of the portfolio.

Example 6.4 (Mixed fractional Black Scholes model). A fractional Brownian motion \( Z \) with Hurst parameter \( H \in (0,1) \) is a centered Gaussian process with covariance

\[
E[Z_t Z_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad t,s \geq 0.
\]

By the Kolmogorov-Centsov criterion, e.g. [19], Theorem 2.2.8, \( Z \) can be chosen \((H - \epsilon)\)-Hölder continuous on compacts for every \( \epsilon < 0 \). In particular \( Z \) can be chosen \( 1/2 \)-Hölder continuous on compacts, if \( H > 1/2 \). The mixed fractional Black-Scholes model is of the form

\[
X_t = X_0 \exp\{\sigma W_t + \eta Z_t + \nu t + \mu t^{2H}\}
\]

for constants \( \sigma, \eta > 0 \) and \( X_0, \mu, \nu \in \mathbb{R} \), where \( W \) is a Brownian motion and \( Z \) is a fractional Brownian motion with Hurst parameter \( H > 1/2 \) independent of \( W \). An application of Corollary 6.2 (taking Propositions 4.7 and 5.6 into account) shows that the mixed fractional Black-Scholes model with \( H > 1/2 \) does not admit simple arbitrage with bounded stopping times. Note that \( X \) is not a semimartingale with respect to its own augmented filtration, if \( 1/2 < H \leq 3/4 \), but is equivalent to the Black-Scholes model for \( H > 3/4 \), see e.g. Cheridito [6].

6.2 Mixed stochastic volatility models

We now generalize Corollary 6.2 from a Brownian motion to stochastic volatility models.

**Theorem 6.5.** Suppose \( (W, B) \) is a two-dimensional Brownian motion with respect to the filtration \( (\mathcal{F}_t) \), and \( Z \) and \( V \) are \( (\mathcal{F}_t) \)-adapted processes such that \( V \) is continuous and \( Z \) is \( 1/2 \)-Hölder continuous on compacts. Assume that \( W \) is independent of \( (B, V, Z) \). Then, for \(-1 < \rho < 1 \) and \( f, g \in \)
\(C([0, T] \times \mathbb{R}) \) such that \(g(t, V_t)\) is strictly positive,

\[
X_t = X_0 \exp \left\{ \int_0^t f(s, V_s)ds + \rho \int_0^t g(s, V_s)dB_s + \sqrt{1-\rho^2} \int_0^t g(s, V_s)dW_s + Z_t \right\}
\]

is free of simple arbitrage with bounded stopping times with respect to the augmentation of the filtration \((\mathcal{F}_t^X)\) generated by \(X\).

Proof. As simple arbitrage is stable with respect to composition with strictly increasing functions, it suffices to show the assertion for \(\log(X_t)\). By Theorem 3.1 in Pakkanen [21], \(\log(X_t)\) satisfies conditional full support with respect to the augmentation of \((\mathcal{F}_t^X)\). Hence (NOA) is satisfied for \(\log(X_t)\).

In view of Corollary 6.1 and Proposition 4.7 it is now sufficient to prove that \((\log(X_t), \mathcal{F}_t)\) satisfies (IUD). We decompose \(\log(X_t) = M_t + Y_t\) with

\[
M_t = \log(X_0) + \rho \int_0^t g(s, V_s)dB_s + \sqrt{1-\rho^2} \int_0^t g(s, V_s)dW_s,
\]

\[
Y_t = Z_t + \int_0^t f(s, V_s)ds.
\]

Then \(M\) is a local martingale with quadratic variation \(\langle M \rangle_t = \int_0^t g^2(s, V_s)ds\). Moreover, \(Y\) is \(1/2\)-Hölder continuous on compacts. Therefore \(M_t + Y_t\) satisfies (IUD) thanks to Theorem 4.1 and Example 4.3.

Example 6.6 (A mixed Heston model). In the Heston model [17] the discounted stock price \(S_t\) has the dynamics

\[
S_t = S_0 \exp \{ \mu t - \frac{1}{2} \int_0^t V_s ds + \rho \int_0^t \sqrt{V_s} dB_s + \sqrt{1-\rho^2} \int_0^t \sqrt{V_s} dW_s \}
\]

\[
V_t = V_0 + \int_0^t \kappa (\theta - V_s) ds + \sigma \int_0^t \sqrt{V_t} dB_s,
\]

where \((W, B)\) is a two-dimensional Brownian motion, \(-1 < \rho < 1, \mu\) is the drift of the discounted stock, \(\theta > 0\) is the long-term limit of the volatility, \(\kappa > 0\) is the mean reversion speed of the volatility and \(\sigma > 0\) is the volatility of volatility. We assume the positivity condition \(2\kappa \theta \geq \sigma^2\) which ensures the strict positivity of \(V_t\). We now define a mixed fractional version of the Heston model by

\[
X_t = S_t e^{Z_t},
\]

where \(Z\) is a fractional Brownian motion with Hurst parameter \(H > 1/2\) (adapted to some filtration with respect to which \((W, B)\) is a two-dimensional Brownian motion) independent of \(W\). Then, by the previous theorem, \(X_t\) does not admit simple arbitrage with bounded stopping times with respect to the augmentation of the filtration \((\mathcal{F}_t^X)\). Of course, the fractional Brownian motion can be replaced by any other \(1/2\)-Hölder continuous processes.
independent of $W$. Moreover, many other stochastic volatility can be cast in the framework of Theorem 6.5 in a similar way as we demonstrated for the Heston model. These include classical stochastic volatility models such as the Hull-White model, the Stein-Stein-model and the Wiggins model (see [20], Ch. 7.4 for more details), but also the model by Barndorff-Nielsen and Shephard [1] with jumps in the volatility and the model by Comte and Renault [8], where volatility is driven by a fractional Brownian motion and exhibits long memory effects. See also the discussion in Section 4 of Pakkanen [21] in the context of conditional full support.

6.3 Mixed local volatility models

Local volatility models were introduced by Dupire [9] in order to capture the smile effect. Here, the stock price $S$ is governed by a SDE

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_0 = s_0,$$

where $W$ is a Brownian motion. Note that the drift $\mu$ and the volatility $\sigma$ depend on time $t$ and the spot price $S_t$. More generally, we will now consider models, where $\mu$ and $\sigma$ may depend on the whole past of the stock price, i.e.

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_0 = s_0$$

(13)

where $\mu, \sigma : [0, \infty) \times C([0, \infty)) \to \mathbb{R}$ are progressive functions satisfying

$$|\mu(t, x)| \leq \bar{\mu} x(t), \quad |\sigma(t, x)| \leq \bar{\sigma} x(t)$$

for some constants $\bar{\mu} > 0$ and $\bar{\sigma} > 0$ for every $t \in [0, \infty)$ and every $x \in C([0, \infty))$ with $x(0) = s_0$. We shall assume that the SDE (13) has a weak solution. It is shown by Pakkanen [21], Section 4.2, that log($S_t$) has conditional full support on $[0, T]$ for every $T > 0$ with respect to the filtration ($\mathcal{F}_t^{S,W}$) generated by $S$ and $W$. We now suppose that a stochastic process $Z$ independent of $(S, W)$ is given, and that $Y$ is 1/2-Hölder continuous on compacts in the sense of Example 4.3, and consider

$$X_t = S_t e^{Z_t}$$

as stock model. Thanks to Remark 5.5 and Propositions 5.4 and 5.6, we can again conclude that $X_t$ satisfies (NOA$_T$) with respect to its own augmented filtration for every $T > 0$. Moreover, by Example 4.3, (ii), it is easy to verify that log($X_t$) satisfies (IUD) with respect to the augmented filtration generated by $(S, W, Z)$ and, hence, with respect to the augmented filtration generated by $X$ (Proposition 4.7). Appealing to Corollary 6.1 we have, thus, proved the following result:

**Theorem 6.7.** Suppose $X_t = S_t e^{Z_t}$, where $S$ is given by (13) and $Z$ is independent of $(S, W)$ and 1/2-Hölder continuous on compacts in the sense of Example 4.3. Then, $X_t$ is free of simple arbitrage with bounded stopping times with respect to the augmentation of the filtration ($\mathcal{F}_t^X$) generated by $X$.
References


