Prof. Dr. Christian Bender
M.Sc. Matthias Thiel

Saarland University
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Stochastics II

11. Tutorial

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ be the underlying filtered probability space for the whole assignment.

Exercise 1 (3 **Points)** Let X be an $\mathbb{F}$-adapted and integrable stochastic process with right-continuous paths. Show that X is a submartingale, if and only if the inequality $\mathrm{E}[X_\sigma] \leq \mathrm{E}[X_\tau]$ holds for all bounded stopping times σ and τ with $\sigma \leq \tau$.

Exercise 2 (3 **Points)** Show that if the stochastic process W is a Brownian motion with respect to the filtration $\mathbb{F}$, then it is also a Brownian motion with respect to $\mathbb{F}_+$.

Exercise 3 (3+4 **Points)** Let $X = (X_t)_{t\in[0,\infty)}$ be a stochastic process with continuous paths, starting at $X_0 = 0$.

(i) Show that X is a Brownian motion with respect to $\mathbb{F}$, if and only if X is $\mathbb{F}$-adapted and
$$\mathrm{E}\left[e^{iu(X_t - X_s)}\middle|\mathcal{F}_s\right] = e^{-\frac{1}{2}u^2(t-s)}$$
for every $0 \leq s < t$ and $u \in \mathbb{R}$.

(ii) Show that X is a Brownian motion with respect to $\mathbb{F}^X$, if and only if X is a Gaussian process with $\mathrm{E}[X_t] = 0$ and $\mathrm{Cov}(X_t, X_s) = t \wedge s$, for $s, t \in [0, \infty)$.

Exercise 4 (4 **Points)** Let $X = (X_t)_{t\in[0,\infty)}$ be a Brownian motion with respect to $\mathbb{F}$ and $a, b > 0$. We take a look at the Gaussian process $Y = (Y_t)_{t\in[0,\infty)}$, given by
$$\forall t \in [0, \infty) : Y_t = \sqrt{a}e^{-bt}X_{e^{2bt}}.$$
Calculate $\mathrm{E}[Y_t]$ and $\mathrm{Cov}(Y_t, Y_s)$ for $s, t \in [0, \infty)$.