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Stochastics II

13. Tutorial

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ be the underlying filtered probability space for the whole assignment.

Exercise 1 (2+3 **Points)**

Let $X := (X_t)_{t\in[0,\infty)}$ a Brownian motion with respect to $\mathbb{F}$ and let τ be an $\mathbb{F}$-stopping time for which one of the following conditions is satisfied,

(i) $(|X_{t\wedge\tau}|)_{t\in[0,\infty)}$ is bounded by a P-integrable random variable,

(ii) $\mathrm{E}(\tau) < \infty$.

Show that $\mathrm{E}[\mathrm{X}_\tau] = 0$.

Hint: To solve (ii), take a look at the random variable

$$\sum_{k=1}^{\lceil\tau\rceil} \max_{t\in[0,1]} |X_{k+t} - X_k|.$$

Exercise 2 (1+2 **Points)** Let τ be a $\mathbb{F}$-stopping time. Show that

(i) $\sigma - \tau$ is a $(\mathcal{F}_{t+\tau})_{t\in[0,\infty)}$-stopping time, if σ is a $\mathbb{F}$-stopping time.

(ii) $\varrho + \tau$ is a $\mathbb{F}$-stopping time, if ϱ is a $(\mathcal{F}_{t+\tau})_{t\in[0,\infty)}$-stopping time.

Exercise 3 (3+4 **Points)** Let $Y = (Y_t)_{t\in[0,T]}$ be a stochastic process defined by

$$Y_t = W_t - \frac{t}{T} W_T,$$

where $(W_t)_{t\in[0,T]}$ is a Brownian motion with respect to $\mathbb{F}' := (\mathcal{F}_t)_{t\in[0,T]}$.

(i) How is the distribution of Y related to the distribution of the Brownian bridge X, from Assignment 12 Exercise 2.

(ii) Calculate the quadratic variation of Y.

Exercise 4 (2 **Points)** Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[0,\infty)$, such that $\frac{\gamma_n}{n} \to 1$ for $n \to \infty$. Show that

$$\lim_{n\to\infty} \sup_{0\le k\le n} \left| \frac{\gamma_k - k}{n} \right| = 0.$$