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WiSe 2018/2019

24th October 2018

Stochastics II

2. Tutorial

Exercise 1 **(4 + 4 Points)**

Let $(\xi, \eta_1, \ldots, \eta_n)$, $n \in \mathbb{N}$ be an $(n+1)$-variate normal distributed random vector, where die random variables $\eta_1, \ldots, \eta_n$ are independend. Calculate

(i) the conditional expectation $\mathrm{E}[\xi|\eta_1, \ldots, \eta_n]$.

(ii) the conditional variance $\mathrm{E}[(\xi - \mathrm{E}[\xi|\eta_1, \ldots, \eta_n)])^2|\eta_1, \ldots, \eta_n]$.

Hint: For the calculation of (i) and (ii) use the random variable

$$(\xi - \mathrm{E}[\xi]) - \sum_{k=1}^{n} \frac{\mathrm{Cov}(\xi, \eta_k)}{\mathrm{Var}(\eta_k)}(\eta_k - \mathrm{E}[\eta_k])$$

and show that it is independent of $\eta - \mathrm{E}[\eta]$, where $\eta := (\eta_1, \ldots, \eta_n)^\top$.

Exercise 2 **(4 Points)** Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathcal{F}, P)$. Show that:

If $\xi_n \geq 0$ and $\liminf_{n\to\infty} \xi_n \in L^1(\mathcal{F}, P)$, then it follows

$$\mathrm{E}[\liminf_{n\to\infty} \xi_n|\mathcal{G}] \leq \liminf_{n\to\infty} \mathrm{E}[\xi_n|\mathcal{G}] \text{ (P-a.s.).}$$

Exercise 3 **(4 Points)** Let X and Y be independent random variables on the probability space $(\Omega, \mathcal{F}, P)$ which map Ω into the measurable spaces $(\mathcal{X}, \mathcal{A})$, respectively $(\mathcal{Y}, \mathcal{G})$. Furthermore let $g : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a measurable function, such that $g(X, Y) \in L^1(\Omega, \mathcal{F}, P)$. Show that:

$$\mathrm{E}[g(X, Y)|X] = \mathrm{E}[g(x, Y)]\big|_{x=X} \ (P_X\text{-a.s.})$$

Exercise 4 **(4 Points)** Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(X_i)_{i\in\mathbb{N}} \subseteq L^1(\Omega, \mathcal{F}, P)$ a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, P)$ and $N \in L^1(\Omega, \mathcal{F}, P)$ a $\mathbb{N}_0$-valued random variable on $(\Omega, \mathcal{F}, P)$ that is independent of $(X_i)_{i\in\mathbb{N}}$. Furthermore set $S_N := \sum_{i=1}^{N} X_i$ with the convention $\sum_{i=1}^{0}(\cdots) := 0$.

(i) Show that S_N is a random variable on $(\Omega, \mathcal{F}, P)$, i.e. show that S_N is $\mathcal{F}$ measurable.

Hint: Represent S_N as a limit of $\mathcal{F}$ measurable functions.

(ii) Find the conditional expectation $\mathrm{E}[S_N|N]$ of S_N given N.