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Stochastics II

3. Tutorial

Exercise 1 (5 Points)

Let X be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ be two σ-algebras. Furthermore we assume that $\sigma(\sigma(X) \cup \mathcal{H})$ is independent of $\mathcal{G}$. Show that the following equation holds true

$$\mathrm{E}[X|\sigma(\mathcal{G} \cup \mathcal{H})] = \mathrm{E}[X|\mathcal{H}].$$

Hint: You may assume that X is a nonnegative(why?). In this case, one can first show that

$$\mathrm{E}[X \mathbb{1}_{A \cap B}] = \mathrm{E}[\mathrm{E}[X|\mathcal{H}] \mathbb{1}_{A \cap B}] \quad A \in \mathcal{G}, B \in \mathcal{H}.$$

In a final step this identity must be extended from sets of the form $A \cap B (A \in \mathcal{G}, B \in \mathcal{H})$ to general events in $\sigma(\mathcal{G} \cup \mathcal{H})$.

Exercise 2 (5 + 2 Points) Let $X := (X_t)_{t \in \mathcal{T}}$ be a stochastic process with state space $(E, \mathcal{E})$.

(i) Show that X is measurable in each of the following situations:
   (a) $\mathcal{T}$ is at most countable.
   (b) $\mathcal{T} = [0, \infty), (E, \mathcal{E}) = (\mathbb{R}^D, \mathcal{B}^D)$ and X has right- or left-continiuous trajectories, where $\mathcal{B}^D$ denotes the Borel σ-field on $\mathbb{R}^D$.

(ii) Give an example for a stochastic process that is not measurable.

Exercise 3 (4 + 2 Points) Let $Y_k := \sum_{j=1}^{k} Z_j, k \in \mathbb{N}$, where $(Z_j)_{1 \leq j \leq k}$ is a independent family of exponential(λ)-distributed random variables with $\lambda > 0$.

(i) Show that Y_k is a Γ-distributed random variable with parameters (k, λ), i.e. the density f_k of Y_k is given by

$$f_k(u) = \frac{(\lambda u)^{k-1}}{(k-1)!} \lambda e^{-\lambda u} \mathbb{1}_{(0,\infty)}(u)$$

for every $u \in \mathbb{R}$.

(ii) Show that

$$P(\{Y_k > \vartheta\}) = \sum_{j=0}^{k-1} \frac{(\lambda \vartheta)^j}{j!} e^{-\lambda \vartheta}$$

for every $\vartheta > 0$.