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Stochastics II

5. Tutorial

Exercise 1 (3 **Points)** Let $N = (N_t)_{t\in[0,\infty)}$ be a Poisson process with intensity $\lambda > 0$. Show that

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda \text{ P-a.s.}$$

Exercise 2 (4 **Points)** Show that a stochastic process $X := (X_t)_{t\in[0,\infty)}$, which is non-decreasing and integrable, has a modification which is P-a.s. rightcontinuous with left limits, if and only if the function $t \mapsto \mathrm{E}[X_t]$ is rightcontinuous.

Exercise 3 (5 **Points)** Let $\lambda : [0,\infty) \to [0,\infty)$ be a function with $\int_s^t \lambda(u)\,du < \infty$ for all $s,t \in \mathbb{R}$. Let

$$\{P_{t_1,\ldots,t_n} : n \in \mathbb{N}, 0 \le t_1 < \cdots < t_n\}$$

be a family of finite dimensional distributions, where $P_{t_1,\ldots,t_n}$ is a probability measure on $(\mathbb{N}_0^n, 2^{\mathbb{N}_0^n})$ for $n \in \mathbb{N}$ and $0 \le t_1 < \cdots < t_n$, which is given by

$$P_{t_1,\ldots,t_n}(\{k_1,\ldots,k_n\}) = \begin{cases} \prod_{j=1}^n \frac{e^{-\lambda_j}\lambda_j^{k_j-k_{j-1}}}{(k_j-k_{j-1})!} & , k_1 \le \cdots \le k_n \\ 0 & , \text{else,} \end{cases}$$

with $k_0 := 0$, $t_0 := 0$, and $\lambda_j = \int_{t_{j-1}}^{t_j} \lambda(u)\,du$ for $j = 1,\ldots,n$. Show that this family of finite dimensional distributions is consistent.

Exercise 4 (5 **Points)** Let

$$\mathcal{P} := \{P_{t_1,\ldots,t_n} : n \in \mathbb{N}, 0 \le t_1 < \cdots < t_n\}$$

be a family of finite dimensional distributions. Show that this family is consistent, if and only if the following identity holds for all $n \in \mathbb{N}$, $0 \le t_1 < \cdots < t_n$ and $j = 1,\ldots,n$:

$$\varphi_{P_{t_1,\ldots,t_n}}(u_1,\ldots,u_{j-1},0,u_{j+1},\ldots,u_{n+1}) = \varphi_{P_{t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n}}(u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_{n+1}).$$

Hint: It could be helpful to define a measure $\tilde{P}$ on $(\mathbb{R}^{n-1}, \mathcal{B}(\mathbb{R}^{n-1}))$ such that

$$\tilde{P}(A_1,\ldots,A_{j-1},A_{j+1},\ldots,A_n) = P_{t_1,\ldots,t_n}(A_1,\ldots,A_{j-1},\mathbb{R},A_{j+1},\ldots,A_n)$$

for $P_{t_1,\ldots,t_n} \in \mathcal{P}$ and all $A_1,\ldots,A_n \in \mathcal{B}(\mathbb{R}^n)$.