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Stochastics II

6. Tutorial

Exercise 1 (6 **Points**) Let $(W_t)_{t\in[0,\infty)}$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$. We define for $n \in \mathbb{N}$ and $(t,x) \in [0,\infty) \times \mathbb{R}$ the recursion

$$\begin{aligned} H_n(t,x) &= H_{n-1}(t,x)x - \frac{\partial}{\partial x} H_{n-1}(t,x)t \\ H_1(t,x) &= x. \end{aligned}$$

Show that the stochastic process $X^{(n)} = (H_n(t,W_t))_{t\in[0,\infty)}$ is an $\mathbb{F}$-martingale for every $n \in \mathbb{N}$.
Hint: Let $\varphi_{0,t-s}(u) := \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{u^2}{2(t-s)}}$ and show that

$$\begin{aligned} &\int H_{n-1}(t,u+y)u\varphi_{0,t-s}(u)du \\ &= t\int \frac{\partial}{\partial u} H_{n-1}(t,u+y)\varphi_{0,t-s}(u)du - s\frac{\partial}{\partial y}\int H_{n-1}(t,u+y)\varphi_{0,t-s}(u)du \end{aligned}$$

Exercise 2 (3 **Points**) Let $X = (X_n)_{n\in\mathbb{N}}$ be a P-integrable stochastic process which is adapted to a filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$. Show that X is an $\mathbb{F}$-martingale if and only if

$$\mathrm{E}[X_{n+1}|\mathcal{F}_n] = X_n$$

P-a.s. for every $n \in \mathbb{N}$.

Exercise 3 (1+2 **Points**) Let $\xi = (\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with

$$P(\{\xi_n = 1\}) = P(\{\xi_n = -1\}) = \frac{1}{2}.$$

Define the stochastic process $M = (M_n)_{n\in\mathbb{N}}$, where

$$M_n = \prod_{j=1}^{n} (1 + \alpha\xi_j)$$

and $\alpha \in (0,1)$ is fixed.

(i) Show that M is an $\mathbb{F}^\xi$-martingale.

(ii) Show that $\lim\limits_{n\to\infty} M_n$ exists P-a.s. and calculate the limit.

Exercise 4 (2 **Points**) Let $(W_t)_{t\in[0,\infty)}$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$. Find $\mu \in \mathbb{R}$ such that the process $(X_t^\mu)_{t\in[0,\infty)}$, with

$$X_t^\mu := \exp\left(W_t - \mu t\right)$$

is an $\mathbb{F}$-martingale.

Exercise 5 (1+3 **Points**)

(i) Let $\mathcal{T} \subset [0,\infty)$, $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$ be a filtration and $\alpha, \beta \in \mathbb{R}$. Show that, if $(X_t)_{t\in\mathcal{T}}$ and $(Y_t)_{t\in\mathcal{T}}$ are $\mathbb{F}$-martingales, then $(\alpha X_t + \beta Y_t)_{t\in\mathcal{T}}$ is also an $\mathbb{F}$-martingale.

(ii) Let $\gamma \in \mathbb{R}$, X_1 and Y_1 be independent random variables with

$$P(\{X_1 = \gamma\}) = P(\{X_1 = -\gamma\}) = P(\{Y_1 = \gamma\}) = P(\{Y_1 = -\gamma\}) = \frac{1}{2}.$$

Furthermore we define the random variable Z by

$$Z := \begin{cases} 1, & X_1 + Y_1 = 0 \\ -1, & X_1 + Y_1 \neq 0 \end{cases}$$

and the random variables $X_2 := X_1 + Z$, and $Y_2 = Y_2 + Z$.
Check wether $X := (X_n)_{n=1,2}$, $Y = (Y_n)_{n=1,2}$ and $X + Y := (X_n + Y_n)_{n=1,2}$ are martingales with respect to the filtration generated by the respective process. Why is this not a counterexample to (i)?