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Stochastics II

7. Tutorial

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ be the underlying filtered probability space for the whole assignment.

Exercise 1 (3+3 **Points**) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function.

(i) Let X be an integrable random variable, such that $f(X) \in L^1(P)$ and $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. Show that,

$$\mathrm{E}[f(X)|\mathcal{G}] \geq f(\mathrm{E}[X|\mathcal{G}]) \ P\text{-a.s.}$$

(ii) Show that $(f(X_t))_{t\in[0,\infty)}$ is a submartingale, if

a) $(X_t)_{t\in[0,\infty)}$ is a martingale and $f(X_t) \in L^1(P)$ for every $t \in [0,\infty)$.

b) $(X_t)_{t\in[0,\infty)}$ is a submartingale, $f(X_t) \in L^1(P)$ for every $t \in [0,\infty)$, and f is non-decreasing.

Hint: Every convex function $f : \mathbb{R} \to \mathbb{R}$ can be expressed as the supremum over all linear functions which lie completely below the graph of f.

Exercise 2 (2 **Points**) Suppose $M = (M_t)_{t\in[0,\infty)}$ is a martingale such that $\mathrm{E}[M_t^2] < \infty$ for every $t \geq 0$. Show that $\mathrm{E}[(M_t - M_s)(M_v - M_u)] = 0$ for every $u \leq v \leq s \leq t$.

Exercise 3 (3 **Points**) Let J be a arbitrary index set and $X := (X_j)_{j\in J} \subset L^1(P)$ be a family of random variables. Show that X is uniformly integrable, if there exists a measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{t\to\infty} \frac{f(t)}{t} = \infty \text{ and } \sup_{j\in J} \mathrm{E}[f(|X_j|)] < \infty$$

Exercise 4 (2 **Points**) Let $X = (X_k)_{k\in\mathbb{N}}$ a sequence of real and independent random variables with $\mathrm{E}[X_k] = 0$ for every $k \in \mathbb{N}$ and $\sum_{k\in\mathbb{N}} \mathrm{E}[X_k^2] < \infty$. Furthermore define the sequence $(M_n)_{n\in\mathbb{N}}$ with $M_n = \sum\limits_{k=1}^{n} X_k$. Show that $(M_n)_{n\in\mathbb{N}}$ converges P-a.s. and in $L^1(P)$ for $n \to \infty$.

Exercise 5 (5 **Points**) Let $W = (W_t)_{t\in[0,\infty)}$ be a Brownian motion and

$$\forall t \in [0,\infty): \quad X_t := \exp\left(W_t - \frac{t}{2}\right).$$

Check if the stochastic process $X := (X_t)_{t\in[0,\infty)}$ converges for $t \to \infty$ P-a.s., respectively in $L^1(P)$ and find the corresponding limit.