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Stochastics II

8. Tutorial

Exercise 1 (6 **Points**) Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ be a filtered probability space and $X = (X_t)_{t\in[0,\infty)}$ be a $\mathbb{F}$-adapted and integrable stochastic process. Show that the following statements are equivalent.

(i) X is a supermartingale.

(ii) For all $s, t \in [0, \infty)$ and every $A \in \mathcal{F}_s$, we have

$$P(A) > 0 \Rightarrow \mathrm{E}^{P(\cdot\,|A)}[X_{s+t}] \leq \mathrm{E}^{P(\cdot\,|A)}[X_s].$$

(iii) For all $s, t \in [0, \infty)$ and every $A \in \mathcal{F}_s$, we have

$$\int_A X_{s+t}\, dP \leq \int_A X_s\, dP.$$

(iv) For all $s, t \in [0, \infty)$ and every bounded, non-negative and $\mathcal{F}_s$ measurable function Y, we have

$$\mathrm{E}[Y X_{s+t}] \leq \mathrm{E}[Y X_s].$$

Exercise 2 (3 **Points**) Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}, P)$ be a filtered probability space, $X = (X_n)_{n\in\mathbb{N}_0}$ be a $\mathbb{F}$-adapted stochastic process, and $a, b \in \mathbb{R}$ such that $a < b$. Define

$$\begin{aligned}\tau_k &:= \min\{n \geq \sigma_{k-1} : X_n \leq a\}, \quad k \in \mathbb{N}\\ \sigma_k &:= \min\{n \geq \tau_k : X_n \geq b\}, \quad k \in \mathbb{N}, \quad \sigma_0 := 0,\end{aligned}$$

and show that τ_k and σ_k are $\mathbb{F}$-stopping times for every $k \in \mathbb{N}$.

Exercise 3 (3 **Points**) We define the graph $Z^2 := (V, E)$, where the set of vertices V and the set of edges E are given by

$$\begin{aligned}V &:= \mathbb{Z}^2\\ E &:= \{\{u, v\} \,|\, u, v \in V,\ \|u - v\|_2 = 1\}.\end{aligned}$$

Here we call $e := \{u, v\}$ the edge between u and v. Now we go through each edge of the grid and decide, by a coin toss, if we remove the edge (heads) or not (tails). Here the coin shows heads with probability p. We denote the resulting random graph by $\tilde{Z}^2$. Show that the probability for the existence of an infinitely large connected component in $\tilde{Z}^2$ is either 1 or 0.

Exercise 4 (4+2 **Points)**

(i) Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}, P)$ be a filtered probability space, $X = (X_n)_{n\in\mathbb{N}}$ be a $\mathbb{F}$-submartingale and σ be a $\mathbb{F}$-stopping time satisfying $\mathrm{E}[\sigma] < \infty$. Furthermore there exists a constant $C > 0$ such that

$$|X_{n+1} - X_n| \leq C$$

P-a.s. for every $n \in \mathbb{N}_0$. Show that

$$\mathrm{E}[X_\sigma] \geq \mathrm{E}[X_0]$$

(ii) Let $(\xi_n)_{n\in\mathbb{N}}$ a sequence of independent random variables with

$$P(\{\xi_n = -1\}) = P(\{\xi_n = 1\}) = \frac{1}{2}.$$

Moreover define $S_n := \sum_{k=1}^{n} \xi_k$ and $\tau := \inf\{n \in \mathbb{N} | S_n = 1\}$. Show that $\mathrm{E}[\tau] = \infty$.