Behavior of the Torsion of the Differential Module of an Algebroid Curve under Quadratic Transformations

Robert W. Berger

Universität des Saarlandes, Naturwiss.-Technische Fakultät I, Fachrichtung Mathematik , Postfach 15 11 50, D-66041 Saarbrücken, Germany.

Abstract. The question, whether the torsion submodule T of the differential module of the local ring R of a singular point of an algebraic or algebroid curve is not zero, is still open in general. All examples suggest the conjecture that this torsion genuinely decreases when going from R to the first quadratic transform R_1, which would imply that T was nontrivial in the first place. We give a general formula for the difference $\ell_R(T) - \ell_{R_1}(T_1)$ of the lengths of these torsions. In the special cases that R is a semigroup ring which is a complete intersection or that R is a "nice" almost complete intersection or a "stable" complete intersection (definition 1) the conjecture is proved.

Introduction

Let R be the local ring of a point of an algebraic or algebroid curve over a field k of characteristic zero, and let $\Omega(R/k)$ denote its universal finite Kähler module of differentials over k. For the sake of simplicity we assume in this paper that R is complete without zero divisors and that k is algebraically closed.
There is a conjecture that the torsion submodule T of $\Omega(R/k)$ is non trivial if R is not regular. The answer to this problem is still open in general, while it is affirmative in a number of special cases. (See for instance [6]. Unfortunately this report contains many misprints which are not the author's fault. A correct version can be obtained from the author via e-mail.)
In [4] we have shown that, if there is a torsion, the canonical homomorphism from $\Omega(R/k)$ into the differential module $\Omega(R_1/k)$ of its first quadratic transform R_1 has a non trivial kernel. Further the torsion is trivial if R is regular. Since R_1 is "less singular" than R it is therefore plausible to conjecture that the length of the torsion will decrease when going from $\Omega(R/k)$ to $\Omega(R_1/k)$.[1]
If one could show that this length genuinely decreases, without previously knowing that there was a non trivial torsion, this would be a means of proving that there was a non trivial torsion.

[1] This does not follow from the fact that there is a non trivial kernel of the map $\Omega(R/k) \to \Omega(R_1/k)$, since in general new torsion elements might arise in $\Omega(R_1/k)$. Also there may be rings between R and R_1 whose differential module has a bigger torsion than that of R, as was shown by J. Heinrich in [8].

If R is a complete intersection there is a simple formula for the length $\ell(T)$. Then R_1 need not be a complete intersection too [2], but if it is, the difference $\ell(T) - \ell(T_1)$ can easily be computed and is > 0 if R is singular (Theorem 2). A complete intersection with this property will be called a "stable" complete intersection. Especially this is the case for a plane curve or a curve through a non singular point of a (twodimensional) surface.
The general case of an arbitrary complete intersection in more complicated. But if R is also a semigroup ring then R_1 is a semigroup ring too, and we can at least give an explicit lower bound for the difference of the two torsions, which is > 0 if R is not regular (Remark 6).
Similarly, if R is an almost complete intersection and R_1 is a complete intersection one has a formula for $\ell(T) - \ell(T_1)$ which shows that this difference is > 0 if R is singular (Theorem 2). In this case we call R a "nice" almost complete intersection.

1 Notations and Remarks

We denote by
k an *algebraically closed* field,
R a *complete* analytic local k-algebra of dimension one without zero divisors,
$\mathfrak{m}$ the maximal ideal of R,
K the quotient field of R,
S the integral closure of R in K.
Then $S \simeq k[\![t]\!]$ is a formal power series ring in a variable t and therefore a discrete valuation ring. Further S is a finitely generated R-module.
Let
ν denote the valuation on K with value group $\mathbb{Z}$ defined by S.
Any $0 \neq x \in \mathfrak{m}$ is a system of parameters for R, and therefore R is a finitely generated module over the discrete valuation ring
$s := k[\![x]\!]$.
Since R has no zero divisors R is a even a free s-module and therefore

[2] In [7, prop.1] P. Carbonne claims that the first quadratic transform of a complete intersection is also a complete intersection. That this is false in general even for semigroup rings can be seen from the following simple example (see section 6 for the notation): Let $R := k[\![t^5, t^8, t^{12}]\!]$ be the semigroup ring belonging to the semigroup generated by $< 5, 8, 12 >$. This is a symmetric semigroup. Since the semigroup is generated by three elements R is a complete intersection ([10, Cor.3.12]). Indeed we have $R \cong k[\![x, X_2, X_3]\!]/(X_2^3 - X_3^2, X_2X_3 - x^4)$. Here $x = t^5$ is a superficial element and $R_1 = k[\![t^3, t^5, t^7]\!]$. This is the semigroup ring belonging to the semigroup generated by $< 3, 5, 7 >$, which is not symmetric. So, again by [10, Cor.3.12] R_1 is not a complete intersection. In fact R_1 is an almost complete intersection:
$R_1 \cong k[\![Z_2, x, Z_3]\!]/(Z_2^3 x - Z_3^2, Z_2Z_3 - x^2, Z_2^4 - xZ_3)$. (The relation $Z_2^4 - xZ_3$ had not been taken into account by [7, prop.1]) For an extensive discussion of numerical semigroups and semigroup rings see also [9].

R is flat over s.
Let
$x \in \mathfrak{m}$ be an element of *minimal value* with respect to ν.
Then
$q := (K : \mathrm{Quot}(s))$ is the *multiplicity of the local ring* R.
$x \in \mathfrak{m} \setminus \mathfrak{m}^2$, and therefore x is part of a minimal system of generators for $\mathfrak{m}$.
Let
$n := \operatorname{edim} R$ be the *embedding dimension* of R and
$\{x, x_2, \ldots, x_n\}$ a minimal system of generators for $\mathfrak{m}$ as an R-module. One can choose the x_i so that $\nu(x) < \nu(x_i)$ for $i = 2, \ldots, n$. Then the first quadratic transform R_1 of R is defined by
$R_1 := R\left[\frac{x_2}{x}, \ldots, \frac{x_n}{x}\right]$
Since $\nu(x) < \nu(x_i)$ for $i = 2, \ldots, n$ we have $R_1 \subseteq S$. Therefore R_1 is also a finitely generated free s-module and a local ring. We denote by
$\mathfrak{m}_1$ the maximal ideal of R_1.
$\mathfrak{m}_1$ is generated by $\left\{x, \frac{x_2}{x}, \ldots, \frac{x_n}{x}\right\}$. R_1 does not depend on the choice of x and the x_i (see [15]).

Further we denote by
$\mathrm{D} : S \longrightarrow \Omega(S/k)$ the universal finite derivation of S over k,
$\mathrm{d}_1 : R_1 \longrightarrow \Omega(R_1/k)$ the universal finite derivation of R_1 over k,
$\mathrm{d} : R \longrightarrow \Omega(R/k)$ the universal finite derivation of R over k,
$\delta_1 : R_1 \longrightarrow \Omega(R_1/s)$ the universal derivation of R_1 over s,
$\delta : R \longrightarrow \Omega(R/s)$ the universal finite derivation of R over s,
$T_1 := \tau(\Omega(R_1/k))$ the torsion submodule of $\Omega(R_1/k)$,
$T := \tau(\Omega(R/k))$ the torsion submodule of $\Omega(R/k)$.
Both these torsion modules have finite length as R_1- and R-modules respectively.
Since k is algebraically closed $R_1/\mathfrak{m}_1 = R/\mathfrak{m} = s/s \cdot x = k$ and therefore for any R_1-, R- or s-module of finite length the lengths ℓ_{R_1}, ℓ_R and ℓ_s as R_1- or R- or s- module respectively are equal to the dimension $\dim_k$ as k-vector space.

2 A formula for the torsion of a complete intersection

Let R be a complete intersection.[3] Represent R as a homomorphic image of the formal power series ring $P := k[\![x, X_2, \ldots, X_n]\!] = s[\![X_2, \ldots, X_n]\!]$.

[3] Let R be a noetherian local Ring, P a regular local ring, $\mathfrak{a}$ an ideal of P, and $R \simeq P/\mathfrak{a}$. Then the minimal number of generators $\mu_P(\mathfrak{a})$ of $\mathfrak{a}$ as an P-module is given by $\mu_P(\mathfrak{a}) = \dim P - \operatorname{edim} R + \varepsilon_1(R)$ (see [14, Th. 21.1] for the formula and the beginning of [14, §21] for the definition of the invariant $\varepsilon_1(R)$.) One needs at least $\dim P - \dim R$ generators. So, in general, one has $\mu_P(\mathfrak{a}) = \dim P - \dim R + d$ with a non negative integer d. In fact, d is an invariant, the "deviation", of R and therefore independent of the choice of P, because from $\dim P - \dim R + d = \mu_P(\mathfrak{a}) =$

Let $\varphi : s[\![X_2,\ldots,X_n]\!] \longrightarrow R$ with $\varphi|s = \mathrm{id}$ and $\varphi(X_i) := x_i$ for $i = 2,\ldots,n$ be that representation.
Then, since P is a regular local ring of dimension n and R is a complete intersection of dimension 1, the kernel $\mathfrak{a}$ of φ is generated by $n-1$ elements.
By [3], Satz 7 we have for an almost complete intersection
$\ell_R(T) = \ell_R(S \,\mathrm{D}\, S/R \,\mathrm{D}\, R) + \ell_R(S/R) + \ell_R\bigl(\mathfrak{D}_K(R/s)^{-1}/R^*\bigr)$,
where $\mathfrak{D}_K$ denotes the Kähler different and R^* complementary module of R over s. The classical Dedekind different $\mathfrak{D}_D(R/s)$ is defined by
$\mathfrak{D}_D(R/s) := R^{*-1}$. If R is a *complete* intersection then R is a Gorenstein ring and therefore $\mathfrak{D}_D(R/s)^{-1} = (R^{*-1})^{-1} = R^*$, so that the formula can be written as
$\ell_R(T) = \ell_R(S \,\mathrm{D}\, S/R \,\mathrm{D}\, R) + \ell_R(S/R) + \ell_R(\mathfrak{D}_K(R/s)^{-1}/\mathfrak{D}_D(R/s)^{-1})$.
By [11], Satz 1 $\mathfrak{D}_K(R/s)$ is equal to the Noether different $\mathfrak{D}_N(R/s)$, and by [1], III, Satz 7 the Noether different $\mathfrak{D}_N(R/s)$ is equal to $\mathfrak{D}_D(R/s)$. Therefore $\mathfrak{D}_K(R/s) = \mathfrak{D}_D(R/s)$. (See also [13] Corollary G.12.)
So be obtain:

Theorem 1. *If R is a complete intersection then*

$$\ell_R(T) = \ell_R(S \,\mathrm{D}\, S/R \,\mathrm{D}\, R) + \ell_R(S/R)$$

Remark 1. If R is a complete intersection which is also a semigroup ring then by Corollary 2 we have $\ell_R(S \,\mathrm{D}\, S/R \,\mathrm{D}\, R) = \ell_R(S/R)$, so that in this case

$$\ell_R(T) = 2 \cdot \ell_R(S/R)$$

(For the case of a plane curve see also [19], Theorem 4)

3 Nice almost complete intersections and stable complete intersections

Definition 1. R is called a *nice almost complete intersection* if R is an almost complete intersection, and its first quadratic transform R_1 is a complete intersection.
R is called a *stable complete intersection* if R and also its first quadratic transform R_1 are both complete intersections.

$\dim P - \operatorname{edim} R + \varepsilon_1(R)$ follows:

$$d = d(R) = \varepsilon_1(R) - \bigl(\operatorname{edim} R - \dim R\bigr)\,.$$

In the case $d(R) = 0$ the local ring R is called a “complete intersection”.
In the case $d(R) = 1$ the local ring R is called an “almost complete intersection”.
While complete intersections are Gorenstein rings this is not so for almost complete intersections (see [12]).
If one chooses $P := s[\![X_2,\ldots,X_n]\!]$, then $\dim P = n = \operatorname{edim} R$ and therefore $\mu_S(\mathfrak{a}) = \varepsilon_1(R)$.

Example 1. The local ring R of a point of a curve on a surface in a non singular point of the surface (e.g. a plane curve) is always a stable complete intersection:
The maximal ideal $\mathfrak{m}$ of R is generated by two elements $\{x, x_2\}$. Then the maximal ideal $\mathfrak{m}_1$ of R_1 is also generated by two elements$\{x, \frac{x_2}{x}\}$. It follows that both rings are factor rings of formal power series rings in two variables. Since their dimensions are one, their relation ideals are principal, and so both rings are complete intersections.

If R is a nice almost complete intersection we can apply the formula of [3], Satz 7 to R and the formula of Theorem 1 to R_1, obtaining:

$$\begin{array}{c} S \,\mathrm{D}\, S \\ | \\ R_1 \,\mathrm{D}\, R_1 \\ | \\ R \,\mathrm{D}\, R \end{array}$$

$\ell_R(T) \;= \ell_R(S \,\mathrm{D}\, S/R \,\mathrm{D}\, R) \;\; + \ell_R(S/R) + \ell_R\big(\mathfrak{D}_K(R/s)^{-1}/R^*\big),$
$\ell_{R_1}(T_1) = \ell_{R_1}(S \,\mathrm{D}\, S/R_1 \,\mathrm{D}\, R) + \ell_{R_1}(S/R_1),$
while in the case of a stable complete intersection the first formula is reduced to
$\ell_R(T) = \ell_R(S \,\mathrm{D}\, S/R \,\mathrm{D}\, R) + \ell_R(S/R).$

On the other hand, since $\ell_R = \dim_k = \ell_{R_1}$, subtracting the two formulas above we get:

Theorem 2. *If R is a nice almost complete intersection then*

$$\ell_R(T) - \ell_{R_1}(T_1) = \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R) + \dim_k(R_1/R) + \dim_k\big(\mathfrak{D}_K(R/s)^{-1}/R^*\big)$$

If R is a stable complete intersection then

$$\ell_R(T) - \ell_{R_1}(T_1) = \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R) + \dim_k(R_1/R)$$

Remark 2. If R is also a semigroup ring then by Corollary 3 we have
$\dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R) = \dim_k(R_1/R)$ so that in this case we obtain
$\ell_R(T) - \ell_R(T_1) = 2\dim_k(R_1/R) + \dim_k\big(\mathfrak{D}_K(R/s)^{-1}/R^*\big)$
in the case of a nice almost complete intersection and
$\ell_R(T) - \ell_{R_1}(T_1) = 2\dim_k(R_1/R)$
in the case of a stable complete intersection, where the last formula follows already by applying Remark 1 to R and R_1.

4 A general formula for $\ell_R(T) - \ell_{R_1}(T_1)$

Since K is separable algebraic and finite over Quot(s) we have $\mathrm{D}\, x \neq 0$. By Satz 6 of [3] applied to R_1 and to R over s respectively and using the fact that $\ell_R = \dim_k = \ell_{R_1}$ we get

$$\ell_R(T) \;= \dim_k \Omega(R/s) - \dim_k \Omega(S/s) \; - \underbrace{\big(\dim_k(S/R) - \dim_k(S \,\mathrm{D}\, S/R \,\mathrm{D}\, R)\big)}_{\geq 0 \text{ by Corollary 2}}.$$

$$\ell_{R_1}(T_1) = \dim_k \Omega(R_1/s) - \dim_k \Omega(S/s) - \underbrace{\big(\dim_k(S/R_1) - \dim_k(S \,\mathrm{D}\, S/R_1 \,\mathrm{D}\, R_1)\big)}_{\geq 0 \text{ by Corollary 2}}.$$

Subtracting both formulas we obtain:

Proposition 1.

$$\ell_R(T) - \ell_{R_1}(T_1) =$$
$$\dim_k \Omega(R/s) - \dim_k \Omega(R_1/s) - \big(\dim_k(R_1/R) - \dim_k(R_1 \operatorname{D} R_1/R \operatorname{D} R)\big)\,.$$

The above mentioned formulas for $\ell_R(T)$, $\ell_{R_1}(T_1)$ and $\ell_R(T) - \ell_{R_1}(T_1)$ can be rewritten in the following way:
From the obvious inclusions $\operatorname{D} R \subseteq R \operatorname{D} R \subseteq S \operatorname{D} S$ one gets
$\dim_k(S \operatorname{D} S/R \operatorname{D} R) = \dim_k(S \operatorname{D} S/\operatorname{D} R) - \dim_k(R \operatorname{D} R/\operatorname{D} R)$. By Corollary 2 one has $\dim_k(S \operatorname{D} S/\operatorname{D} R) = \dim_k(S/R)$. Therefore
$\dim_k(S/R) - \dim_k(S \operatorname{D} S/R \operatorname{D} R) = \dim_k(R \operatorname{D} R/\operatorname{D} R)$.
By the same argument:
$\dim_k(S/R_1) - \dim_k(S \operatorname{D} S/R_1 \operatorname{D} R_1) = \dim_k(R_1 \operatorname{D} R_1/\operatorname{D} R_1)$.
Substituting these expressions in the above formulas for $\ell_R(T)$ and $\ell_{R_1}(T_1)$ and then computing the difference yields:

Proposition 2.
$\ell_R(T) \;= \dim_k \Omega(R/s) - \dim_k \Omega(S/s) \;- \dim_k(R \operatorname{D} R/\operatorname{D} R)\,.$
$\ell_{R_1}(T_1) = \dim_k \Omega(R_1/s) - \dim_k \Omega(S/s) - \dim_k(R_1 \operatorname{D} R_1/\operatorname{D} R_1)\,.$
$\ell_R(T) - \ell_{R_1}(T_1) =$
$$\dim_k \Omega(R/s) - \dim_k \Omega(R_1/s) - \big(\dim_k(R \operatorname{D} R/\operatorname{D} R) - \dim_k(R_1 \operatorname{D} R_1/\operatorname{D} R_1)\big)\,.$$

Definition 2. The expressions
$\operatorname{Ed}(R) := \dim_k(R \operatorname{D} R/\operatorname{D} R)$ and $\operatorname{Ed}(R_1) := \dim_k(R_1 \operatorname{D} R_1/\operatorname{D} R_1)$ will be called the *"exactness-defects"* of R and R_1 respectively.

Remark 3. The exactness-defect is a measure for the number of non exact differentials in the ring. For semigroup rings the exactness-defects are zero (Proposition 4). Rings with exactness-defect zero have "maximal torsion" ([16, Th. 1]). For an interpretation of $\Omega(R/k)/\operatorname{d} R$ as a cyclic homology module see [18, HC_1 9.6.15]. In the graded case one obtains the Kernel of the canonical map $\Omega(R/k) \to \Omega(S/k)$ as shown in [17, Th.2.1], which in our case is $T = \tau\big(\Omega(R/k)\big)$.

Remark 4. If R is a semigroup ring then by Corollary 3 we have
$\dim_k(R_1 \operatorname{D} R_1/R \operatorname{D} R) = \dim_k(R_1/R)$ so that in this case we have

$$\ell_R(T) - \ell_{R_1}(T_1) = \dim_k \Omega(R/s) - \dim_k \Omega(R_1/s)\,.$$

Using the techniques already employed in [4] we will now give an **estimate for the difference $\dim_k \Omega(R/s) - \dim_k \Omega(R_1/s)$:**
Let $z_i := \frac{x_i}{x}$ for $i = 2 \ldots n$. Write R_1 and R as factor rings of polynomial rings over s in Z_i and X_i with relation ideals $\mathfrak{n}_1$ and $\mathfrak{n}$ respectively:
$R_1 \simeq P_1/\mathfrak{n}_1$ with $P_1 := s[Z_2, \ldots, Z_n]$

$R \simeq P/\mathfrak{n}$ with $P := s[X_2, \ldots, X_n]$.
One has a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{n}_1 \arrow[r] & P_1 \arrow[r, "Z_i \mapsto z_i"] & R_1 \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[u] & P \arrow[r, "X_i \mapsto x_i"'] \arrow[u, "X_i \mapsto xZ_i"'] & R \arrow[r] \arrow[u, "x_i \mapsto xz_i"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

In the following we identify P with its image in P_1, so that we have $X_i = x \cdot Z_i$ for all $i = 2, \ldots, n$.
Now let
$\Delta_1 : P_1 \to F_1 := \Omega(P_1/s)/\mathfrak{n}_1 \cdot \Omega(P_1/s)$ and
$\Delta : P \to F := \Omega(P/s)/\mathfrak{n} \cdot \Omega(P/s)$ denote the compositions of the universal derivations of P_1 and P over s with the factor maps mod $\mathfrak{n}_1 \cdot \Omega(P_1/s)$ and $\mathfrak{n} \cdot \Omega(P/s)$ respectively. Then
$F_1 = R_1 \cdot \Delta Z_2 \oplus \cdots \oplus R_1 \cdot \Delta Z_n$ and
$F = R \cdot \Delta X_2 \oplus \cdots \oplus R \cdot \Delta X_n$
are the free modules with bases ΔZ_i and ΔX_i respectively and from the above diagram one obtains an exact commutative diagram of R-modules:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{N}_1 \arrow[r] & F_1 \arrow[r, "\Delta_1 Z_i \mapsto \delta_1 z_i"] & \Omega(R_1/s) \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{N} \arrow[r] \arrow[u] & F \arrow[r, "\Delta X_i \mapsto \delta x_i"'] \arrow[u, "\Delta X_i \mapsto x\Delta_1 Z_i"'] & \Omega(R/s) \arrow[r] \arrow[u, "\delta x_i \mapsto x\delta_1 z_i"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

where $\mathfrak{N}_1 := R_1 \cdot \Delta_1 \mathfrak{n}_1$ and $\mathfrak{N} := R \cdot \Delta\mathfrak{n}$ are the corresponding relation modules (e.g. see [2], Satz 5).
Since s is a principal ideal domain and both R_1 and R are finitely generated s-modules they are also free s-modules, and their rank as s-modules is equal to $q = (K : \mathrm{Quot}(s)$. Therefore F_1 and F are both free s-modules of rank $q \cdot (n-1)$.
Since $F_1/\mathfrak{N}_1 \simeq \Omega(R_1/s)$ and $F/\mathfrak{N} \simeq \Omega(R/s)$ are both s-modules of finite length, the ranks of $\mathfrak{N}_1$ and $\mathfrak{N}$ as s-modules are the same as the ranks of F_1 and F, namely $q \cdot (n-1)$.
From now on we identify F with its image in F_1 We have
$F_1 = \quad R_1 \cdot \Delta_1 Z_2 \oplus \cdots \oplus \quad R_1 \cdot \Delta_1 Z_n$
$F \ = R \cdot x \cdot \Delta_1 Z_2 \oplus \cdots \oplus R \cdot x \cdot \Delta_1 Z_n$

and therefore

$$\begin{aligned}\ell_s(F_1/F) = (n-1)\cdot\ell_s(R_1/x\cdot R) = \\ (n-1)\cdot\ell_s(R_1/R) + (n-1)\cdot\ell_s(R/x\cdot R) \\ = (n-1)\cdot\ell_s(R_1/R) + (n-1)\cdot q,\end{aligned}$$

because R is a free s-module of rank q, and so $\ell_s(R/x\cdot R) = q\cdot\underbrace{\ell_s(s/x\cdot s)}_{=1} = q$

On the other hand we see from the above diagram that:
$\ell_s(F_1/F) + \ell_s(\underbrace{F/\mathfrak{N}}_{\simeq\Omega(R/s)}) = \ell_s(\underbrace{F_1/\mathfrak{N}_1}_{\simeq\Omega(R_1/s)}) + \ell_s(\mathfrak{N}_1/\mathfrak{N})$.
It follows:

$$\ell_s(\Omega(R/s)) - \ell_s(\Omega(R_1/s)) = \ell_s(\mathfrak{N}_1/\mathfrak{N}) - (n-1)\ell_s(R_1/R) - (n-1)\cdot q \quad (1)$$

Using the fact that $\ell_s = \dim_k$ and substituting equation (1) into Proposition 1 we obtain

$$\begin{aligned}\ell_R(T) - \ell_{R_1}(T_1) = \\ \dim_k(R_1 \mathrm{D}\, R_1/R \,\mathrm{D}\, R) + \ell_s(\mathfrak{N}_1/\mathfrak{N}) - (n-1)\cdot q - n\cdot\dim_k(R_1/R) \quad (2)\end{aligned}$$

We now try to evaluate $\ell_s(\mathfrak{N}_1/\mathfrak{N})$.
Let $h\in\mathfrak{n}$ be an arbitrary element of $\mathfrak{n}$. Then $h = \sum\limits_{i_2,\dots,i_n}\alpha_{i_2,\dots,i_n}X_2^{i_2}\dots X_n^{i_n}$ with $\alpha_{i_2,\dots,i_n}\in s$. Since $h\in\mathfrak{n}$ we have $\sum\limits_{i_2,\dots,i_n}\alpha_{i_2,\dots,i_n}x_2^{i_2}\dots x_n^{i_n} = 0$. It follows that $\alpha_{0,\dots,0}\in\mathfrak{m}\cap s = x\cdot s$. Therefore we can write
$h = \beta_1 x + \beta_2 X_2 + \dots + \beta_n X_n + \sum\limits_{i_2+\dots+i_n\geq 2}\alpha_{i_2,\dots,i_n}X_2^{i_2}\dots X_n^{i_n}$.
$\{x, x_2, \dots, x_n\}$ is a *minimal* system of generators for $\mathfrak{m}$ as an R-module. Therefore the residue classes of $x, x_2, \dots, x_n$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $R/\mathfrak{m}$. Therefore $\beta_1,\dots,\beta_n\in\mathfrak{m}\cap s = x\cdot s$. So h is of the form.
$h(X_2,\dots,X_n) = x^2\cdot\gamma_1 + x\cdot\sum\limits_{i=2}^{n}\gamma_i X_i + \sum\limits_{i_2+\dots+i_n\geq 2}\alpha_{i_2,\dots,i_n}X_2^{i_2}\dots X_n^{i_n}$ with $\gamma_i\in s$.
Substituting $X_i = x_i\cdot Z_i$ it follows that there is an $f\in s[Z_2,\dots,Z_n]$ with $h(X_2,\dots,X_n) = h(x\cdot Z_2,\dots,x\cdot Z_n) = x^2\cdot f(Z_2,\dots,Z_n)$.
Since P_1 has no zero divisors f is uniquely determined by h, and we write $f = \frac{1}{x^2}h$.

$$\begin{array}{c}\mathfrak{n}_1\\ \big|\\ \tilde{\mathfrak{n}}\\ \big|\\ \mathfrak{n} = x^2\cdot\tilde{\mathfrak{n}}\end{array}$$

Because of $x_i = x\cdot z_i$ for $i = 2,\dots,n$ we have $x^2\cdot f(z_2,\dots,z_n) = h(x_2,\dots,x_n) = 0$, and therefore $f(z_2,\dots,z_n) = 0$, which shows that $f(Z_2,\dots,Z_n)\in\mathfrak{n}_1$.
Define $\tilde{\mathfrak{n}} := \{\frac{1}{x^2}h \mid h\in\mathfrak{n}\}$. Then $\mathfrak{n}_1\supseteq\tilde{\mathfrak{n}}\supset\mathfrak{n} = x^2\cdot\tilde{\mathfrak{n}}$.
Obviously $\tilde{\mathfrak{n}}$ is a P-module, and if $\mathfrak{n}$ is generated by

$\{h_1, \ldots, h_r\}$ as a P-ideal then $\tilde{\mathfrak{n}}$ is generated by the corresponding $\{\frac{1}{x^2}h_1, \ldots, \frac{1}{x^2}h_r\}$ as a P-module.

Denote by $\widetilde{\mathfrak{N}} := R \cdot \Delta_1 \tilde{\mathfrak{n}}$ the R-submodule of $\mathfrak{N}_1$ generated by $\{\Delta_1 f \mid f \in \tilde{\mathfrak{n}}\}$. Then we have $\mathfrak{N} = R \cdot \Delta\mathfrak{n} = R \cdot \Delta_1(x^2 \cdot \tilde{\mathfrak{n}}) = x^2 \cdot \widetilde{\mathfrak{N}}$.
Since $\mathfrak{N}$ is a free s-module of rank $(n-1) \cdot q$ then so is $\widetilde{\mathfrak{N}} = \frac{1}{x^2} \cdot \mathfrak{N}$, and we get

$$\begin{tikzcd}[row sep=small] \mathfrak{N}_1 = R_1 \cdot \Delta_1 \mathfrak{n}_1 \arrow[d, dash] \\ \widetilde{\mathfrak{N}} = R \cdot \Delta_1 \tilde{\mathfrak{n}} \arrow[d, dash] \\ \mathfrak{N} = R \cdot \Delta\mathfrak{n} = x^2 \cdot \widetilde{\mathfrak{N}} \end{tikzcd}$$

$$\boldsymbol{\ell_s(\widetilde{\mathfrak{N}}/\mathfrak{N})} = \ell_s(\widetilde{\mathfrak{N}}/x^2 \cdot \widetilde{\mathfrak{N}}) = (n-1) \cdot q \cdot \underbrace{\ell_s(s/x^2 \cdot s)}_{=2} \ \boldsymbol{= 2(n-1) \cdot q.}$$

Then
$\ell_s(\mathfrak{N}_1/\mathfrak{N}) = \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) + \ell_s(\widetilde{\mathfrak{N}}/\mathfrak{N}) = \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) + 2 \cdot (n-1) \cdot q$.
Substituting this into equation (1) we get

$$\begin{aligned} &\ell_s(\Omega(R/s)) - \ell_s(\Omega(R_1/s)) = \\ &\qquad \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) + (n-1) \cdot q - (n-1) \cdot \dim_k(R_1/R) \end{aligned} \tag{3}$$

and therefore with equation (2):

Theorem 3.

$$\begin{aligned} &\ell_R(T) - \ell_{R_1}(T_1) = \\ &\quad \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) + \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R) + (n-1) \cdot q - n \cdot \dim_k(R_1/R) \\ &\qquad = \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) - (n-1) \cdot \dim_k(R_1/R) + (n-1) \cdot q \\ &\qquad\qquad - (\dim_k(R_1/R) - \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R)) \end{aligned}$$

Remark 5. If R is a semigroup ring then by Corollary 3 we have $\dim_k(R_1 \,\mathrm{d}\, R_1/R \,\mathrm{D}\, R) = \dim_k(R_1/R)$, so that in this case

$$\begin{aligned} \ell_R(T) - \ell_{R_1}(T_1) &= \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) - (n-1) \cdot \dim_k(R_1/R) + (n-1) \cdot q \\ &= \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) - (n-1) \cdot \big(\dim_k(R_1/R) - q\big), \end{aligned}$$

which follows also from Remark 4 together with formula 3.

5 $\boldsymbol{\ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}})}$ for a complete intersection

Proposition 3. *If the projective dimension* **p-dim** $\boldsymbol{\Omega(R/s) \leq 1}$ *then*

$$\ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) = \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}_1) + (n-1) \cdot \dim_k(R_1/R) \geq (n-1) \cdot \dim_k(R_1/R),$$

where $\widetilde{\mathfrak{N}}_1 := R_1 \cdot \widetilde{\mathfrak{N}}$.

Proof. $\Omega(R/s) = F/\mathfrak{N}$, where F is a free R-module of rank $n-1$. Now p-dim$(F/\mathfrak{N}) \leq 1$ so that $\mathfrak{N}$ is also a free R-module, and since $\ell_R(F/\mathfrak{N}) < \infty$ the rank of $\mathfrak{N}$ as R-module is equal to the rank of F. $\mathfrak{N}$ is generated by the Δh with $h \in \mathfrak{n}$. Since R is local one can choose a basis of the free R-module $\mathfrak{N}$ of the form $\{\Delta h_1, \ldots, \Delta h_{n-1} \mid h_i \in \mathfrak{n} \text{ for all i}\}$. Let $f_i := \frac{1}{x^2} h_i$ be the corresponding generators of $\tilde{\mathfrak{n}}$. Then

$$\begin{array}{l}
\mathfrak{N}_1 = R_1 \cdot \Delta_1 \mathfrak{n}_1 \\
\quad | \\
\widetilde{\mathfrak{N}}_1 = R_1 \cdot \Delta_1 \tilde{\mathfrak{n}} = R_1 \cdot \widetilde{\mathfrak{N}} \\
\quad | \\
\widetilde{\mathfrak{N}} = R \cdot \Delta_1 \tilde{\mathfrak{n}} \\
\quad | \\
\mathfrak{N} = R \cdot \Delta \mathfrak{n} = x^2 \cdot \widetilde{\mathfrak{N}}
\end{array}$$

$\{\Delta_1 f_1, \ldots, \Delta_1 f_{n-1}\}$ is a system of generators of $\widetilde{\mathfrak{N}}$ as an R-module and also an R-basis because both modules have the same rank as R-modules since $\mathfrak{N} = x^2 \cdot \widetilde{\mathfrak{N}}$. Let
$\widetilde{\mathfrak{N}}_1 := R_1 \cdot \widetilde{\mathfrak{N}}$ be the R_1-submodule of F_1 generated by $\widetilde{\mathfrak{N}}$.
Then $\{\Delta_1 f_1, \ldots, \Delta_1 f_{n-1}\}$ is a basis of $\widetilde{\mathfrak{N}}_1$ as an R_1-module and therefore
$\widetilde{\mathfrak{N}}_1/\widetilde{\mathfrak{N}} \simeq \bigoplus_{i=1}^{n-1} R_1/R,$
from which we get:
$\boldsymbol{\ell_s(\widetilde{\mathfrak{N}}_1/\widetilde{\mathfrak{N}}) = (n-1) \cdot \dim_k(R_1/R)}$.
Further $\mathfrak{N}_1 \supseteq \widetilde{\mathfrak{N}}_1 \supseteq \widetilde{\mathfrak{N}}$, and therefore
$\ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) = \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}_1) + \ell_s(\widetilde{\mathfrak{N}}_1/\widetilde{\mathfrak{N}}) = \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}_1) + (n-1) \cdot \dim_k(R_1/R)$

If R is a complete intersection, $\mathfrak{n}$ is generated by $n-1$ elements $\{h_1, \ldots, h_{n-1}\}$. Then $\mathfrak{N}$ is generated by $\{\Delta h_1, \ldots, \Delta h_{n-1}\}$ and therefore p-dim$(F/\mathfrak{N}) = 1$. Then from Theorem 3 and the above Proposition we get:

Corollary 1. *Let R be a complete intersection. Then*

$$\begin{aligned}
&\ell_R(T) - \ell_{R_1}(T_1) \\
&\quad = \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}_1) + (n-1) \cdot q - \big(\dim_k(R_1/R) - \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R)\big) \\
&\qquad\qquad \geq (n-1) \cdot q - \big(\dim_k(R_1/R) - \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R)\big)\,.
\end{aligned}$$

Remark 6. If R is complete intersection and also a semigroup ring then by Corollary 3 we have $\dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R) = \dim_k(R_1/R)$, so that in this case

$$\ell_R(T) - \ell_{R_1}(T_1) = \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}_1) + (n-1) \cdot q \geq (n-1) \cdot q$$

For a semigroup ring which is a *stable* complete intersection we know already from Remark 2 that $\ell_R(T) - \ell_{R_1}(T_1) = 2 \cdot \dim_k(R_1/R)$.

6 Semigroup Rings

Consider a subring R' of S with $k \subseteq R' \subseteq S$ and the universal finite derivation $\mathrm{D} : S \to \Omega(S/k) = S \,\mathrm{D}\, S = S \cdot \mathrm{D}\, t$ of S over k. The Kernel of D is k, because

$\mathrm{D}\big(\sum_{\nu=0}^{\infty}\alpha_\nu\cdot t^\nu\big)=\sum_{\nu=0}^{\infty}\nu\cdot\alpha_\nu\cdot t^{\nu-1}=0\Longleftrightarrow\alpha_\nu=0$ for all $\nu>0$.

D is a k-linear map, and since $\operatorname{Ker}\mathrm{D}=k\subseteq R'$ we have $\mathrm{D}^{-1}(\mathrm{D}\,R')=R'$. So we get:

Remark 7. Let $k\subseteq R'\subseteq R''\subseteq S$ be two subrings of S. Then the universal finite derivation D of S over k induces an Isomorphism of k-vectorspaces

$$R''/R' \underset{k}{\simeq} \mathrm{D}\,R''/\,\mathrm{D}\,R'$$

Now let $R':=k[\![t^{n_1},\dots,t^{n_r}]\!]\subseteq S=k[\![t]\!],\quad n_1<\cdots<n_r$,
be a semigroup ring. We show that in this case $R'\,\mathrm{D}\,R'=\mathrm{D}\,R'$:

Proposition 4.
Let R' be a semigroup ring. Then:
$\mathrm{D}:R'\to R'\,\mathrm{D}\,R'$ *is surjective, i.e.* $R'\,\mathrm{D}\,R'=\mathrm{D}\,R'$. *(Each differential of* $R'\,\mathrm{D}\,R'$ *is exact with respect to* $\mathrm{D}\,|R'$.*)*

Proof. Since $R'\,\mathrm{D}\,R'=R'\cdot\mathrm{D}\,t^{n_1}+\cdots+R'\cdot\mathrm{D}\,t^{n_r}$, it is sufficient to show that each $\omega\in R'\,\mathrm{D}\,t^{n_i}$ has an inverse image in R' $(i=1,\dots,r)$:
$\omega\in R'\cdot\mathrm{D}\,t^{n_i}\Rightarrow$

$$\begin{aligned}
\omega &= \sum_{\nu_1,\dots,\nu_r}\alpha_{\nu_1,\dots,\nu_r}\cdot(t^{n_1})^{\nu_1}\cdots(t^{n_i})^{\nu_i}\cdots(t^{n_r})^{\nu_r}\cdot\mathrm{D}\,t^{n_i}\\
&= \sum_{\nu_1,\dots,\nu_r}\alpha_{\nu_1,\dots,\nu_r}\cdot(t^{n_1})^{\nu_1}\cdots(t^{n_i})^{\nu_i}\cdots(t^{n_r})^{\nu_r}\cdot n_i\cdot t^{n_i-1}\cdot\mathrm{D}\,t\\
&= \sum_{\nu_1,\dots,\nu_r} n_i\cdot\alpha_{\nu_1,\dots,\nu_r}\cdot t^{\nu_1n_1+\cdots+\nu_in_i+n_i-1+\cdots+\nu_rn_r}\cdot\mathrm{D}\,t\\
&= \mathrm{D}\left(\sum_{\nu_1,\dots,\nu_r}\frac{n_i}{\nu_1n_1+\cdots+(\nu_i+1)n_i+\cdots+\nu_rn_r}\alpha_{\nu_1,\dots,\nu_r}\cdot t^{\nu_1n_1+\cdots+(\nu_i+1)n_i+\cdots+\nu_rn_r}\right)\\
&= \mathrm{D}\underbrace{\left(\sum_{\nu_1,\dots,\nu_r}\frac{n_i}{\nu_1n_1+\cdots+(\nu_i+1)n_i+\cdots+\nu_rn_r}\alpha_{\nu_1,\dots,\nu_r}\cdot(t^{n_1})^{\nu_1}\cdots(t^{n_i})^{\nu_i+1}\cdots(t^{n_r})^{\nu_r}\right)}_{\in R'}
\end{aligned}$$

Since $S=k[\![t]\!]$ is a semigroup ring it follows from proposition 4 that
$S\,\mathrm{D}\,S=\mathrm{D}\,S$. From Remark 7 we obtain:

Corollary 2.

$$S\,\mathrm{D}\,S/\,\mathrm{D}\,R\underset{k}{\simeq}S/R\quad\textit{as }k\textit{-vectorspaces}$$

and therefore
$\dim_k(S\,\mathrm{D}\,S/R\,\mathrm{D}\,R)\le\dim_k(S\,\mathrm{D}\,S/\,\mathrm{D}\,R)=\dim_k(S/R).$

$$\begin{array}{ccc}
S & \xrightarrow{\mathrm{D}} & \mathrm{D}\,S=S\,\mathrm{D}\,S\\
| & & |\\
| & & R\,\mathrm{D}\,R\\
| & & |\\
R & \xrightarrow{\mathrm{D}} & \mathrm{D}\,R
\end{array}$$

If R is a semigroup ring. Then:

$$S\,\mathrm{D}\,S/R\,\mathrm{D}\,R\underset{k}{\simeq}S/R\quad\textit{as }k\textit{-vectorspaces}$$

and therefore

$$\dim_k(S\,\mathrm{D}\,S/R\,\mathrm{D}\,R)=\dim_k(S/R).$$

If R is a semigroup ring then also R_1 is a semigroup ring. So we can apply proposition 4 to R and to R_1. It follows with Remark 7:

Corollary 3. *Let R be a semigroup ring. Then:*

$$R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R \underset{k}{\simeq} R_1/R \quad \textit{as k-vectorspaces}$$

and therefore

$$\dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R) = \dim_k(R_1/R).$$

7 Summary

$$\ell_R(T) - \ell_{R_1}(T_1) =$$

General Formula:

$$= \dim_k \Omega(R/s) - \dim_k \Omega(R_1/s) - \big(\dim_k(R_1/R) - \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R)\big)$$
$$= \dim_k \Omega(R/s) - \dim_k \Omega(R_1/s) - \big(\dim_k(R \,\mathrm{D}\, R/\,\mathrm{D}\, R) - \dim_k(R_1 \,\mathrm{D}\, R_1/\,\mathrm{D}\, R_1)\big)$$
$$= \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) + \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R) + (n-1)\cdot q - n \cdot \dim_k(R_1/R)$$

Complete Intersection:

$$= \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}_1) + (n-1)\cdot q - \big(\dim_k(R_1/R) - \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R)\big)$$

Semigroup Ring:

$$= \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}) - (n-1)\cdot \dim_k(R_1/R) + (n-1)\cdot q$$

Stable Complete Intersection:

$$= \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R) + \dim_k(R_1/R)$$

Semigroup Ring, Complete Intersection:

$$= \ell_s(\mathfrak{N}_1/\widetilde{\mathfrak{N}}_1) + (n-1)\cdot q \geq (n-1)\cdot q$$

Semigroup Ring, Stable Complete Intersection:

$$= 2 \cdot \dim_k(R_1/R)$$

Nice Almost Complete Intersection:

$$= \dim_k(R_1 \,\mathrm{D}\, R_1/R \,\mathrm{D}\, R) + \dim_k(R_1/R) + \dim_k(\mathfrak{D}_K(R/s)^{-1}/R^*)$$

References

1. Robert [W.] Berger, *Über verschiedene Differentenbegriffe*, Sitzungsber. Heidelberger Akad. d. Wiss., Math.-naturw. Kl., 1. Abh., 1960, Springer–Verlag, pp. 1–44.
2. ———, *Ausdehnung von Derivationen und Schachtelung der Differente*, Math. Z. **78** (1962), 97–115.
3. ———, *Differentialmoduln eindimensionaler lokaler Ringe*, Math. Z. **81** (1963), 326–354.

4. Robert W. Berger, *On the torsion of the differential module of a curve singularity*, Arch. Math. (Basel) **50** (1988), 526–533.
5. ———, *Algebraische Theorie der Differentiale*, Ann. Univ. Sarav. Ser. Math. **5** (1994), no. 2, 103–244.
6. ———, *Report on the Torsion of the Differential Module of an Algebraic Curve*, Algebraic Geometry and its Applications (Chandrajit L. Bajaj, ed.), Springer-Verlag, 1994, pp. 285–303.
7. Philippe Carbonne, *Sur les différentielles de torsion*, J. Algebra **202** (1998), 367–403.
8. Johannes Heinrich, *Torsion of Differential Modules of Quasihomogeneous Algebroid Curves*, Ann. Univ. Sarav. Ser. Math. **9** (1999), no. 4, 195–204.
9. Jürgen Herzog, *Generators and Relations of Abelian Semigroups and Semigroup Rings*, Ph.D. thesis, Louisiana State University, 1969.
10. ———, *Generators and Relations of Abelian Semigroups and Semigroup Rings*, manus. math. **3** (1970), 175–193, Part of the author's Ph. D. thesis.
11. Ernst Kunz, *Vollständige Durchschnitte und Differenten*, Arch. Math. (Basel) **19** (1968), 47–58.
12. ———, *Almost complete intersections are not Gorenstein rings*, J. Algebra **28** (1974), 111–115.
13. ———, *Kähler Differentials*, Advanced Lectures in Mathematics, Vieweg, Braunschweig/Wiesbaden, 1986.
14. Hideyuki Matsumura, *Commutative Ring Theory*, Cambridge University Press, 1990.
15. D.G. Northcott, *On the notion of a first neighbourhood ring with an application to the $AF+B\Phi$ theorem*, Math. Proc. Cambridge Philos. Soc. **53** (1957), 43–56.
16. Thomas Pohl, *Differential modules with maximal torsion*, Arch. Math. (Basel) **57** (1991), 438–445.
17. Leslie G. Roberts, *Kahler differentials and HC_1 of certain graded K-algebras*, Algebraic K-Theory: Connections with Geometry and Topology (L. F. Jardine and V. P. Snaith, eds.), Kluwer Academic Publishers, 1989, pp. 389–424.
18. Charles A. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics, vol. 38, Cambridge University Press, 1995.
19. Oscar Zariski, *Characterization of plane algebroid curves whose module of differentials has maximum torsion*, Proc. Nat. Acad. Sci. **56** (1966), no. 3, 781–786, Erratum p. 1927.

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