Algebraic Geometry and Commutative Algebra. Summer term 2024 Prof. Dr. S. Brandhorst



Exercise Sheet 1

All rings are commutative with 1, unless stated otherwise.

Exercise 1

- (a) Let k be an infinite field and $f \in k[x_1, \ldots x_n]$ a non-zero polynomial. Show that there exists a point $a \in \mathbb{A}^n(k)$ such that $f(a) \neq 0$.
- (b) Let \mathbb{F}_2 be the field of two elements and $f = xy(x+y) \in \mathbb{F}_2[x,y]$. Show that

$$\mathbb{A}^2(\mathbb{F}_2) \to \mathbb{F}_2, \quad (a_1, a_2) \mapsto f(a_1, a_2)$$

is the zero map.

Exercise 2 Let R be a ring and $I, J, P \leq R$ ideals. Show the following statements.

(a)
$$\sqrt{\sqrt{I}} = \sqrt{I}$$

(b) $\sqrt{I} = (1) \iff I = (1)$

(c)
$$\sqrt{I+J} = \sqrt{\sqrt{I} + \sqrt{J}}$$

- (d) if P is prime, then $\sqrt{P^n} = P$ for all n > 0.
- (e) The Jacobson radical J(R) of R is the intersection of all maximal ideals of R. Show that $x \in J(R) \iff 1 xy$ is a unit in R for all $y \in R$.
- (f^{*}) Show that a radical ideal I is the intersection of all prime ideals containing it. (Hint: Zorn's Lemma)

Exercise 3

- (a) Show that a ring R is Noetherian if and only if every non-empty set of ideals of R contains a maximal element.
- (b) Let R be a Noetherian ring. Show that every ideal I of R is finitely generated, i.e. there exist $r_1, \ldots r_n \in R$ with $I = (r_1, \ldots r_n)$.
- (c) Let R be a ring such that every ideal of R is finitely generated. Show that R is Noetherian.

Exercise 4

- (a) Let R be a Noetherian ring and let $I \subseteq R$ be an ideal. Show that the quotient ring R/I is also Noetherian.
- (b) If R is a Noetherian ring and if $f: R \to S$ is a surjective homomorphism onto a ring S, show that S is also Noetherian.