Algebraic Geometry and Commutative Algebra.
Summer term 2024
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Exercise Sheet 1

All rings are commutative with 1, unless stated otherwise.

Exercise 1

(a) Let k be an infinite field and $f \in k[x_1, \dots x_n]$ a non-zero polynomial. Show that there exists a point $a \in \mathbb{A}^n(k)$ such that $f(a) \neq 0$.

(b) Let $\mathbb{F}_2$ be the field of two elements and $f = xy(x+y) \in \mathbb{F}_2[x,y]$. Show that

$$\mathbb{A}^2(\mathbb{F}_2) \to \mathbb{F}_2, \quad (a_1, a_2) \mapsto f(a_1, a_2)$$

is the zero map.

Exercise 2 Let R be a ring and $I, J, P \leq R$ ideals. Show the following statements.

(a) $\sqrt{\sqrt{I}} = \sqrt{I}$

(b) $\sqrt{I} = (1) \iff I = (1)$

(c) $\sqrt{I+J} = \sqrt{\sqrt{I} + \sqrt{J}}$

(d) if P is prime, then $\sqrt{P^n} = P$ for all $n > 0$.

(e) The Jacobson radical $J(R)$ of R is the intersection of all maximal ideals of R. Show that $x \in J(R) \iff 1 - xy$ is a unit in R for all $y \in R$.

(f*) Show that a radical ideal I is the intersection of all prime ideals containing it. (Hint: Zorn's Lemma)

Exercise 3

(a) Show that a ring R is Noetherian if and only if every non-empty set of ideals of R contains a maximal element.

(b) Let R be a Noetherian ring. Show that every ideal I of R is finitely generated, i.e. there exist $r_1, \dots r_n \in R$ with $I = (r_1, \dots r_n)$.

(c) Let R be a ring such that every ideal of R is finitely generated. Show that R is Noetherian.

Exercise 4

(a) Let R be a Noetherian ring and let $I \subseteq R$ be an ideal. Show that the quotient ring R/I is also Noetherian.

(b) If R is a Noetherian ring and if $f\colon R \to S$ is a surjective homomorphism onto a ring S, show that S is also Noetherian.