Algebraic Geometry and Commutative Algebra.
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Exercise Sheet 10

Exercise 1 Show that the following sets are projective varieties:

(a) the incidence correspondence

$$\{(L, a) \in G(l, n) \times \mathbb{P}^{n-1} : L \subset \mathbb{P}^{n-1} \text{ an } l-1 \text{ dimensional linear subspace with } a \in L\}$$

(b) the join of two disjoint varieties $X, Y \subseteq \mathbb{P}^n$, i.e. the union in $\mathbb{P}^n$ of all lines intersecting X and Y. (Hint: Use that the image of a projective variety under a regular map is closed.)

Exercise 2 Let $R \subseteq S$ be an integral ring extension, i.e. every element of R is integral over S. Let $J \leq S$ be an ideal. We consider $R/(J \cap R)$ as a subring of S/J. Show that S/J is integral over $R/(J \cap R)$.

Exercise 3 Let $A \subseteq B$ be rings and B integral over A. Let $P_1 \subset P_2 \subset P_3 \subset A$ be a chain of prime ideals of A and Q_1, Q_3 be prime ideals in B with $Q_1 \cap A = P_1$ and $Q_3 \cap A = P_3$. Suppose that A is a finitely generated algebra over a field k.

(a) Show that we can find a prime ideal $Q_2 \subset B$ with $Q_1 \subset Q_2 \subset Q_3$.

(b) can we require $Q_2 \cap A = P_2$?

Exercise 4 Let R be a factorial ring and $P \subset R$ a minimal prime ideal. Show that P is principal.