Algebraic Geometry and Commutative Algebra.
Summer term 2024
Prof. Dr. S. Brandhorst

UNIVERSITÄT DES SAARLANDES

Exercise Sheet 11

Exercise 1 Let X be an irreducible affine variety. Show the following:

(a) The function field $K(X)$ is isomorphic to the quotient field $Q(A(X))$ of the coordinate ring $A(X)$ of X.

(b) Every local ring $\mathcal{O}_{X,a}$ for $a \in X$ is naturally a subring of $K(X)$.

Exercise 2

(a) Let $X, Y \subseteq \mathbb{A}^n$ be pure dimensional affine varieties. Show that every irreducible component of $X \cap Y$ has dimension at least $\dim X + \dim Y - n$ (Hint: use the diagonal)

(b) Let $X, Y \subseteq \mathbb{P}^n$ be non-empty projective varieties. Use (a) to show that if $\dim X + \dim Y \geq n$, then $X \cap Y \neq \emptyset$.

(c) Use (b) to show that $\mathbb{P}^{m+n}$ is birational to $\mathbb{P}^m \times \mathbb{P}^n$ but not isomorphic to it.

Exercise 3 Recall that the blowup of $\mathbb{A}^n$ in the origin O is defined as

$$V = \{(x_1, \dots, x_n), (t_1 : \cdots : t_n) \in \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k \mid x_i t_j - x_j t_i = 0, 1 \leq i, j \leq n\}.$$

Let $\pi \colon V \to \mathbb{A}^n$ be the projection to the first factor. If W is an affine variety, then the blowup $\widetilde{W}$ of W in O is defined as the Zariski closure of $\pi^{-1}(W \setminus \{O\})$ in V. Let $\varphi \colon \widetilde{W} \to W$ be the restriction of π. To blow up any other point P move it to the origin by a linear change of coordinates.

(a) let $E = \pi^{-1}(\{O\})$. Describe E and show that π restricts to an isomorphism $V \setminus E \to \mathbb{A}^n \setminus \{O\}$.

(b) Compute the blowup of $W = Z(x^{n+1} + y^2 + z^2)$, $n \geq 1$ in the origin. Is it smooth? What is $\varphi^{-1}(\{O\})$?

Exercise 4 Let $\widetilde{\mathbb{A}}^3$ be the blow-up of $\mathbb{A}^3$ in the line $V(x_1, x_2) \cong \mathbb{A}^1$. Show that its exceptional set is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$. When do the strict transforms of two lines in $\mathbb{A}^3$ through $V(x_1, x_2)$ intersect in the blow-up? What is therefore the geometric meaning of the points in the exceptional set?