Algebraic Geometry and Commutative Algebra.
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Exercise Sheet 9

Exercise 1 Let $Q = Z(x_0x_3 - x_1x_2) \subseteq \mathbb{P}^3_k$, $H = Z(x_0)$ and $P_0 = [1 : 0 : 0 : 0] \in Q$. Consider the rational map

$$\pi \colon \mathbb{P}^3 \dashrightarrow H, \quad [x_0 : x_1 : x_2 : x_3] \mapsto [0 : x_1 : x_2 : x_3],$$

and its restriction $p = \pi|_Q$ whose inverse is

$$q \colon H \dashrightarrow Q, \quad [0 : x_1 : x_2 : x_3] \mapsto \left[\frac{x_1x_2}{x_3} : x_1 : x_2 : x_3\right].$$

The domain of q is $H \setminus S$ where $S = \{[0 : 1 : 0 : 0], [0 : 0 : 1 : 0]\}$.

(a) Let $P \in Q$ be a point. Show that $p(P)$ is the point of intersection of the line $\overline{PP_0}$ with H.

(b) Pick (any) point $P \in H \setminus S$, and compute the intersection $\overline{PP_0} \cap Q$. What is different for $P \in S$? Explain geometrically why q is not defined at points in S.

Exercise 2 Let X and Y be irreducible varieties.

(a) Show that X and Y are birational if and only if their function fields $K(X)$ and $K(Y)$ are isomorphic.

(b) Show that X is birational to a hypersurface. Hint: Use Noether normalisation, and the theorem of the primitive element for field extensions. For simplicity you may assume that the ground field has characteristic zero.

Exercise 3 Let $f \colon X \to Y$ be a morphism of varieties and $a \in X$. Show that it induces a linear map $f_* \colon T_aX \to T_{f(a)}Y$ between tangent spaces.

Exercise 4 (a) Let $f \in K[x_0, \dots, x_n]$ be a homogeneous polynomial of degree d. Show that $\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = d \cdot f$.

(b) Let $X \subseteq \mathbb{P}^n$ be a projective variety, $I(X) = (f_1, \dots f_r)$ and $a \in X$. Prove that X is smooth in a if and only if the rank of the $r \times (n+1)$ Jacobian matrix $J_a = (\frac{\partial f_i}{\partial x_j}(a))_{i,j}$ is at least $n - \operatorname{codim}_X\{a\}$. Note that the entries of the Jacobian matrix J_a depend on the choice of a representative of a. However, its rank is independent of this choice.