Algebraic Geometry and Commutative Algebra.
Summer term 2024
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Exercise Sheet 2

Exercise 1

(a) Show that a ring R is Noetherian if and only if every non-empty set of ideals of R contains a maximal element.

(b) Let R be a Noetherian ring. Show that every ideal I of R is finitely generated, i.e. there exist $r_1, \ldots r_n \in R$ with $I = (r_1, \ldots r_n)$.

(c) Let R be a ring such that every ideal of R is finitely generated. Show that R is Noetherian.

Exercise 2

(a) Let R be a Noetherian ring and let $I \subseteq R$ be an ideal. Show that the quotient ring R/I is also Noetherian.

(b) If R is a Noetherian ring and if $f \colon R \to S$ is a surjective homomorphism onto a ring S, show that S is also Noetherian.

Exercise 3 (a) Let $f \colon X \to Y$ be a continuous map and $V \subseteq X$ an irreducible subset. Show that $f(X)$ is irreducible.

(b) Let X be an irreducible topological space and $U \subseteq X$ be a non-empty open subset. Show that U is dense in X and U is irreducible in its induced topology.

(c) Let $Y \subseteq X$ be a subset which is irreducible in the induced topology. Show that its closure $\overline{Y}$ is irreducible.

Exercise 4 Identify $\mathbb{A}^1 \times \mathbb{A}^1$ with $\mathbb{A}^2$ in the natural way. Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topology on the two copies of $\mathbb{A}^1$.

Exercise 5 A topological space X is said to be disconnected if there are two disjoint open subsets $U_1, U_2 \subseteq X$ with $X = U_1 \cup U_2$. Suppose now that X, U_1 and U_2 are affine varieties. Show that $A(X) = A(U_1) \times A(U_2)$.