Algebraic Geometry and Commutative Algebra. Summer term 2024 Prof. Dr. S. Brandhorst



## Exercise Sheet 2

## Exercise 1

- (a) Show that a ring R is Noetherian if and only if every non-empty set of ideals of R contains a maximal element.
- (b) Let R be a Noetherian ring. Show that every ideal I of R is finitely generated, i.e. there exist  $r_1, \ldots r_n \in R$  with  $I = (r_1, \ldots r_n)$ .
- (c) Let R be a ring such that every ideal of R is finitely generated. Show that R is Noetherian.

## Exercise 2

- (a) Let R be a Noetherian ring and let  $I \subseteq R$  be an ideal. Show that the quotient ring R/I is also Noetherian.
- (b) If R is a Noetherian ring and if  $f: R \to S$  is a surjective homomorphism onto a ring S, show that S is also Noetherian.
- **Exercise 3** (a) Let  $f: X \to Y$  be a continuous map and  $V \subseteq X$  an irreducible subset. Show that f(V) is irreducible. Hint: First reduce to the case that V = X and f is surjective.
- (b) Let X be an irreducible topological space and  $U \subseteq X$  be a non-empty open subset. Show that U is dense in X and U is irreducible in its induced topology.
- (c) Let  $Y \subseteq X$  be a subset which is irreducible in the induced topology. Show that its closure  $\overline{Y}$  is irreducible.

**Exercise 4** Let  $X_1$  and  $X_2$  be two topological spaces. The product topology on  $X_1 \times X_2$  is it is the coarsest topology such that the projections  $X_1 \times X_2 \to X_i$ , i = 1, 2 are continuous. The open subsets of the form  $U_1 \times U_2$  for  $U_i$  open in  $X_i$ , i = 1, 2 form a basis for the product topology.

Identify  $\mathbb{A}^1 \times \mathbb{A}^1$  with  $\mathbb{A}^2$  in the natural way. Show that the Zariski topology on  $\mathbb{A}^2$  is not the product topology of the Zariski topology on the two copies of  $\mathbb{A}^1$ . Hint: Consider the diagonal  $\Delta = \{(x, x) \in \mathbb{A}^2 \mid x \in \mathbb{A}^1\}$ .

**Exercise 5** A topological space X is said to be disconnected if there are two disjoint open subsets  $U_1, U_2 \subseteq X$  with  $X = U_1 \cup U_2$ . Suppose now that  $X, U_1$  and  $U_2$  are affine varieties. Show that  $A(X) = A(U_1) \times A(U_2)$ .