Algebraic Geometry and Commutative Algebra.
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Exercise Sheet 4

Exercise 1 Let R be a ring $I \leq R$ an ideal and $S \subseteq R$ a multiplicatively closed subset.

(a) Show that the equivalence relation used to define the localisation of R at S is indeed transitive.

(b) Show that $S^{-1}R$ is the zero ring if and only if $0 \in S$.

(c) Show that the ring homomorphism $i \colon R \to S^{-1}R$, $x \mapsto x/1$ is injective if and only if S does not contain any zero divisors

(d) Reprove that the radical of an ideal I is the intersection of the prime ideals containing it. To do so consider R/I and suitable localisations of it, use the fact that every nonzero ring contains a prime ideal.

(e) Recall the definition of an R-module M. Generalise the concept of localisation to R-modules and define the $S^{-1}R$-module $S^{-1}M$.

(f) Show that $S^{-1}(R/I) \cong S^{-1}R/S^{-1}I$.

Exercise 2 Let $X \subseteq \mathbb{A}^n$ be an affine variety and let $a \in X$. Show that $\mathcal{O}_{X,a} \cong \mathcal{O}_{\mathbb{A}^n,a}/I(X)\mathcal{O}_{\mathbb{A}^n,a}$ where $I(X)\mathcal{O}_{\mathbb{A}^n,a}$ denotes the ideal in $I(X)\mathcal{O}_{\mathbb{A}^n,a}$ generated by all quotients $f/1$ for $f \in I(X)$.

Exercise 3 Let $I, J, K, I_\lambda, J_\lambda \leq R$, $\lambda \in \Lambda$ be ideals. The ideal quotient of I by J is

$$(I : J) = \{x \in R \mid xJ \subseteq I\}$$

Show the following

(a) $I \subseteq (I : J)$

(b) $((I : J) : K) = (I : (JK)) = ((I : K) : J)$

(c) $((\bigcap_\lambda I_\lambda) : J) = \bigcap_\lambda (I_\lambda : J)$

(d) $(I : \sum_\lambda J_\lambda) = \bigcap_\lambda (I : J_\lambda)$

Exercise 4 Let $Y, Z \subseteq X$ be subvarieties of the affine variety X. Show that

$$\overline{Z \setminus Y} = V(I(Z) : I(Y)).$$