

Exercise Sheet 5

Exercise 1 Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime. Here $\otimes_{\mathbb{Z}}$ means that we take their tensor product as \mathbb{Z} -modules.

Exercise 2 Let R be a ring.

(a) Let $f: M' \rightarrow M$ be an R -module homomorphism and N an R -module. Show that $f^*: \text{Hom}(M, N) \rightarrow \text{Hom}(M', N)$, $f^*(g) = g \circ f$ is an R -module homomorphism.

(b) Let

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \rightarrow 0$$

be a sequence of R modules and homomorphisms. Then this sequence is exact if and only if for all R -modules N the sequence

$$0 \rightarrow \text{Hom}(M'', N) \xrightarrow{v^*} \text{Hom}(M, N) \xrightarrow{u^*} \text{Hom}(M', N)$$

is exact.

(c) Let $u: M' \rightarrow M$ be injective. Show that u^* is not necessarily surjective.

Exercise 3 Let R be a ring, J an ideal contained in the Jacobson radical of R and M an R -module and N a finitely generated R -module, and let $u: M \rightarrow N$ be a homomorphism. If the induced homomorphism $M/JM \rightarrow N/JN$ is surjective, then u is surjective.

Exercise 4 Let $f: X \rightarrow Y$ be a morphism of affine varieties and $f^*: A(Y) \rightarrow A(X)$ the corresponding pullback on coordinate rings. Are the following true or false?

(a) f is surjective if and only if f^* is injective

(b) f is injective if and only if f^* is surjective

(c) If $f: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is an isomorphism then f is affine linear, i.e. of the form $f(x) = ax + b$ for some $a, b \in K$

(d) If $f: \mathbb{A}^2 \rightarrow \mathbb{A}^2$ is an isomorphism then f is affine linear, i.e. of the form $f(x) = Ax + b$ for some $A \in K^{2 \times 2}$ and $b \in K^2$.