Algebraic Geometry and Commutative Algebra.
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Exercise Sheet 5

Exercise 1 Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime. Here $\otimes_{\mathbb{Z}}$ means that we take their tensor product as $\mathbb{Z}$-modules.

Exercise 2 Let R be a ring.

(a) Let $f\colon M' \to M$ be an R-module homomorphism and N an R-module. Show that $f^*\colon \operatorname{Hom}(M, N) \to \operatorname{Hom}(M', N)$, $f^*(g) = g \circ f$ is an R-module homomorphism.

(b) Let

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$$

be a sequence of R modules and homomorphisms. Then this sequence is exact if and only if for all R-modules N the sequence

$$0 \to \operatorname{Hom}(M'', N) \xrightarrow{v^*} \operatorname{Hom}(M, N) \xrightarrow{u^*} \operatorname{Hom}(M', N)$$

is exact.

(c) Let $u\colon M' \to M$ be injective. Show that u^* is not necessarily surjective.

Exercise 3 Let R be a ring, J an ideal contained in the Jacobson radical of R and M an R-module and N a finitely generated R-module, and let $u\colon M \to N$ be a homomorphism. If the induced homomorphism $M/JM \to N/JN$ is surjective, then u is surjective.

Exercise 4 Let $f\colon X \to Y$ be a morphism of affine varieties and $f^*\colon A(Y) \to A(X)$ the corresponding pullback on coordinate rings. Are the following true or false?

(a) f is surjective if and only if f^* is injective

(b) f is injective if and only if f^* is surjective

(c) If $f\colon \mathring{A}^1 \to \mathring{A}^1$ is an isomorphism then f is affine linear, i.e. of the form $f(x) = ax + b$ for some $a, b \in K$

(d) If $f\colon \mathring{A}^2 \to \mathring{A}^2$ is an isomorphism then f is affine linear, i.e. of the form $f(x) = Ax + b$ for some $A \in K^{2\times 2}$ and $b \in K^2$.