Algebraic Geometry and Commutative Algebra.
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Exercise Sheet 6

Exercise 1 Let P, N, be submodules of the R-module M and $S \subseteq R$ a multiplicative subset. We regard $S^{-1}N$ as a submodule of $S^{-1}M$. Show that

(a) $S^{-1}(N+P) = S^{-1}N + S^{-1}P$,

(b) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$

(c) Let $I \subseteq R$ be an ideal. Show that $\sqrt{S^{-1}I} = S^{-1}\sqrt{I}$.

(d) Show that (b) is false for infinite intersections.

Exercise 2 Let R be a ring and $S, T \subseteq R$ multiplicatively closed subsets.
Let $ST = \{st \mid s \in S, t \in T\}$ and let M, N, P be R-modules. Show the following statements.

(a) $(ST)^{-1}M \cong S^{-1}T^{-1}M$,

(b) $S^{-1}M \cong S^{-1}R \otimes_R M$.

(c) $(M \oplus N) \otimes_R P \cong (M \otimes_R P) \oplus (N \otimes_R P)$.

Exercise 3 (a) Prove (or believe) the following Lemma: Let A, B be rings, let M be an A-module, P a B-module and N an (A, B)-bimodule (that is, N is simultaneously an A module and a B-module and the two structures are compatible in the sense that $a(xb) = (ax)b$ for all $a \in A, b \in B, x \in N$). Then $M \otimes_A N$ is naturally a B-module $N \otimes_B P$ an A module and we have

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$$

as (A, B)-bimodule. (Hint: Mimick the proof that $U \otimes_k (V \otimes_k W) \cong (U \otimes_k V) \otimes_k W$ for k-vector spaces U, V, W.)

(b) Use the Lemma and 2 (b) to prove the following important statement: Let R be a ring $S \subseteq R$ a multiplicatively closed subset. Furthermore let M and N be two R-modules. Then

$$S^{-1}(M \otimes_R N) \cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N$$

Exercise 4 Show that the notation $\otimes$ is ambiguous in the following sense:
Find a ring R, two R-modules M, N, submodules $M' \leq M$ and $N' \leq N$ and elements $n' \in N'$ and $m' \in M'$ such that $n' \otimes m'$ is non zero as an element of $M' \otimes_R N'$ but zero as an element of $M \otimes_R N$.