Algebraic Geometry and Commutative Algebra.
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Exercise Sheet 8

Exercise 1 Let S be a graded ring.

(a) Let $S \neq 0$. Show that S is an integral domain if and only if for any two homogeneous elements $f, g \in S$ with $fg = 0$ we have $f = 0$ or $g = 0$.

(b) Prove that a homogeneous ideal $I < S$ is prime if and only if for any two *homogeneous* $f, g \in S$: $fg \in I$ implies $f \in I$ or $g \in I$.

(c) Show that a projective variety is irreducible if and only if its homogeneous coordinate ring is an integral domain.

Exercise 2 (a) Let $\varphi\colon \mathbb{P}^1_k \to \mathbb{P}^2_k$, $\varphi(t_0 : t_1) = [t_0^2 : t_0t_1 : t_1^2]$ and $Y = \varphi(\mathbb{P}^1_k)$. Show that φ is an isomorphism, but the homogeneous coordinate rings $S(\mathbb{P}^1_k)$ and $S(Y)$ are not isomorphic.

(b) Let X, Y be prevarieties. Show that a morphism $f\colon X \to Y$ is an isomorphism if and only if f is a homeomorphism and for all $P \in X$, $f^* : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is an isomorphism. (Hint: First reduce to the case that X and Y are affine varieties)

Exercise 3 (The d-Uple Embedding) For given $n, d > 0$, let $M_0, \dots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \dots, x_n$, where $N = \binom{n+d}{n} - 1$. Define the d-uple embedding by

$$\rho_d\colon \mathbb{P}^n \to \mathbb{P}^N, \quad \rho_d(P) = (M_0(P) : \cdots : M_N(P)).$$

(a) Let J be the kernel of $\rho_d^*\colon k[y_0, \dots y_N] \to k[x_0, \dots x_n], \quad y_i \mapsto M_i$. Show that J is a homogeneous prime ideal.

(b) Show that the image of ρ_d is exactly $Z(J) \subseteq \mathbb{P}^n$.

(c) Show that ρ_d is an isomorphism onto its image.

Exercise 4 If $V(f) = H \subseteq \mathbb{P}^n$ is any hypersurface show that its complement $\mathbb{P}^n \setminus H$ is affine. To this end let d be the degree of (the defining equation f of) H and consider the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$. Then use the fact that the complement of a hyperplane in $\mathbb{P}^N$ is affine.