Funktionentheorie

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Sommersemester 2020

Field of complex numbers

Complex numbers: $\mathbb{C} = \mathbb{R}^2 = \{(a, b); a, b \in \mathbb{R}\}$
Field with respect to the algebraic operations

$$(a, b) + (c, d) = (a + c, b + d), \qquad (a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

Real numbers become a subfield via: $\mathbb{R} = \mathbb{R} \times \{0\} \subset \mathbb{C}$
Introducing the complex notation $i = (0, 1)$ one obtains the rules $i^2 = -1$ and

- $(a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1)(b, 0) = a + ib,$
- $(a + ib)(c + id) = (ac - bd, ad + bc) = (ac - bd) + i(ad + bc),$
- $\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}.$

Definition

For $z = x + iy$ $(x, y \in \mathbb{R})$ we define

- $\mathrm{Re}\, z = x, \qquad \mathrm{Im}\, z = y$
- $\overline{z} = x - iy$
- $|z| = (x^2 + y^2)^{1/2}$

Absolute value in $\mathbb{C}$

Lemma

For $z, w \in \mathbb{C}$, we have:

- $z\bar{z} = |z|^2$, $|z| = |\bar{z}|$, $|zw| = |z||w|$, $\overline{(\bar{z})} = z$,
- $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z} \cdot \bar{w}$, $\overline{(1/z)} = 1/\bar{z}$ *for* $z \neq 0$,
- $\mathrm{Re} z = \frac{z+\bar{z}}{2}$, $\mathrm{Im} z = \frac{z-\bar{z}}{2i}$,
- $\big||z| - |w|\big| \leq |z+w| \leq |z| + |w|$ (**triangle inequality**).

Idea Triangle inequality:

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w}) = |z|^2+|w|^2+2\mathrm{Re}(z\bar{w}) \leq |z|^2+|w|^2+2|z\bar{w}| = (|z|+|w|)^2.$$

Lemma (Absolute value as a norm)

For $z, w \in \mathbb{C}$, we have:

- $|z| \geq 0$,
- $|z| = 0$ *if and only if* $z = 0$,
- $\big||z| - |w|\big| \leq |z+w| \leq |z| + |w|$.

Polar coordinates and arguments

Theorem (Polar coordinates)

For $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, there are real numbers $r > 0, \varphi \in \mathbb{R}$ with

$$z = r(\cos\varphi + i\sin\varphi).$$

In this case $r = |z|$

Idea Set $x = \mathrm{Re}\, z, y = \mathrm{Im}\, z$. Because of

$$|x|, |y| \leq |z| \neq 0$$

and the intermediate value theorem there is a $\psi \in [0, \pi]$ with $x = |z| \cos\psi$. Show that

$$z = r(\cos\varphi + i\sin\varphi) \quad \text{with } \varphi = \psi \text{ or } \varphi = -\psi$$

Corollary (Arguments)

For $z \in \mathbb{C}^*$ and $\theta_0 \in \mathbb{R}$, there is a uniqe $\theta \in [\theta_0, \theta_0 + 2\pi[$ with

$$z = |z|(\cos\theta + i\sin\theta).$$

Idea The existence follows from the preceding theorem and 2π-periodicity of cos, sin, uniqueness from the functional equations for cos, sin.

Arguments and Euler's formula

Definition

(a) *For $z \in \mathbb{C}^*$ and $\theta_0 \in \mathbb{R}$, the unique number $\theta \in [\theta_0, \theta_0 + 2\pi[$ with*

$$z = |z|(\cos\theta + i\sin\theta)$$

is called the argument of z relative to θ_0 (written: $\arg_{\theta_0}(z)$).

(b) *For $\varphi \in \mathbb{R}$, we define $e^{i\varphi} = \cos\varphi + i\sin\varphi$.*

Using the defintion of the product in $\mathbb{C}$ and the functional equations for cos, sin one obtains:

Lemma

For $\varphi, \psi \in \mathbb{R}$, we have

$$e^{i(\varphi+\psi)} = e^{i\varphi}e^{i\psi}.$$

Convergence in $\mathbb{C}$

Let $(z_n)_{n\geq 0}$ be a sequence in $\mathbb{C}$ and let $z \in \mathbb{C}$.

Definition

- $(z_n)_{n\geq 0}$ converges to z (written: $\lim_{n\to\infty} z_n = z$)

$$:\Leftrightarrow \quad \forall \epsilon > 0 \; \exists n_0 \in \mathbb{N} \text{ such that } |z_n - z| < \epsilon \; \forall n \geq n_0$$

- $(z_n)_{n\geq 0}$ is a Cauchy sequence

$$:\Leftrightarrow \quad \forall \epsilon > 0 \; \exists n_0 \in \mathbb{N} \text{ such that } |z_n - z_m| < \epsilon \forall n, m \geq n_0$$

Using the estimates

$$\max(|\mathrm{Re}\, z_n - \mathrm{Re}\, z|, |\mathrm{Im}\, z_n - \mathrm{Im}\, z|) \leq |z_n - z| \leq |\mathrm{Re}\, z_n - \mathrm{Re}\, z| + |\mathrm{Im}\, z_n - \mathrm{Im}\, z|$$

one obtains:

Theorem

- $\lim_{n\to\infty} z_n = z \;\Leftrightarrow\; \lim_{n\to\infty} \mathrm{Re}\, z_n = \mathrm{Re}\, z$ *and* $\lim_{n\to\infty} \mathrm{Im}\, z_n = \mathrm{Im}\, z$*,*
- $(z_n)_{n\geq 0}$ *is a Cauchy sequence in* $\mathbb{C} \;\Leftrightarrow\; (\mathrm{Re}\, z_n)_{n\geq 0}$ *and* $(\mathrm{Im}\, z_n)_{n\geq 0}$ *are Cauchy sequences in* $\mathbb{R}$*.*

Sequences and series

Using the previous theorem and the completeness of $\mathbb{R}$ one obtains:

Each Cauchy sequence in $\mathbb{C}$ converges (In short: $\mathbb{C}$ is complete).

Exactly as in $\mathbb{R}$ one proves:

Theorem (Limit theorems)

Suppose that $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$ in $\mathbb{C}$. Then

- $\lim_{n\to\infty}(z_n + w_n) = z + w, \quad \lim_{n\to\infty}(z_n w_n) = zw$
- *If $w \neq 0$ then there is an $n_0 \in \mathbb{N}$ with $w_n \neq 0 \forall n \geq n_0$. In this case*

$$\left(\frac{z_n}{w_n}\right)_{n\geq n_0} \xrightarrow{n} \frac{z}{w}.$$

Since $\mathbb{C}$ is complete, the Cauchy criterion for series remains true:

$$\sum_{n=0}^{\infty} c_n \text{ converges in } \mathbb{C} \Leftrightarrow \forall \epsilon > 0 \exists n_0 \in \mathbb{N} \text{ with } |\sum_{n=p}^{q} c_n| < \epsilon \forall q \geq p \geq n_0.$$

With exactly the same proof as in $\mathbb{R}$ one concludes that

Lemma

- *Each absolutely convergent series in $\mathbb{C}$ converges*
- *The comparison test and the ratio test remain true for series in $\mathbb{C}$.*

$\mathbb{C}$ as a metric space

$\mathbb{C} = \mathbb{R}^2$ is a metric space relative to the **Euclidean metric**:

$$d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}, \quad d(z, w) = |z - w|$$

Open and closed d-balls of radius $r \in [0, \infty]$:

$$D_r(a) = \{z \in \mathbb{C}; |z - a| < r\}, \qquad \overline{D}_r(a) = \{z \in \mathbb{C}; |z - a| \leq r\}.$$

Definition

Let $A, U \subset \mathbb{C}$.

- *U is open $\Leftrightarrow \forall a \in U \exists \epsilon > 0$ with $D_\epsilon(a) \subset U$*
- *A is closed $\Leftrightarrow \mathbb{C} \setminus A$ is open*
- *$A \subset \mathbb{C}$ is compact if $\forall (U_i)_{i \in I}$ open cover of A $\exists i_1, \ldots, i_r \in I$ with*

$$A \subset U_{i_1} \cup \ldots \cup U_{i_r}.$$

Closure, interior, boundary

Definition

- $\overline{A} = \{z \in \mathbb{C}; \forall \epsilon > 0: \; D_\epsilon(z) \cap A \neq \emptyset\}$
- $\mathrm{Int}(A) = \{z \in \mathbb{C}; \exists \epsilon > 0 \textit{ mit } D_\epsilon(z) \subset A\}$
- $\partial A = \{z \in \mathbb{C}; \forall \epsilon > 0: \; D_\epsilon(z) \cap A \neq \emptyset \neq D_\epsilon(z) \cap A^c\}$

Lemma

- $U \subset \mathbb{C}$ *is open* $\Leftrightarrow$ $U = \mathrm{Int}(U)$
- $A \subset \mathbb{C}$ *is closed* $\Leftrightarrow A = \overline{A}$
- $\mathrm{Int}(A) = \bigcup(U \subset \mathbb{C} \textit{ open}; U \subset A) = A \cap (\partial A)^c \subset \mathbb{C}$ *is open*
- $\overline{A} = \bigcap(F \subset \mathbb{C} \textit{ closed}; F \supset A) = \{z \in \mathbb{C}; \exists \textit{ sequence } A \ni z_n \xrightarrow{n} z\}$ *is closed*
- $\partial A = \overline{A} \setminus \mathrm{Int}(A) \subset \mathbb{C}$ *is closed*
- $A \subset \mathbb{C}$ *is compact* $\Leftrightarrow$ *A is closed and bounded* $\Leftrightarrow \forall$ *sequence* $(z_n)_{n \geq 0}$ *in A* $\exists$ *convergent subsequence* $(z_{n_k}) \xrightarrow{k} z \in A$.

Paths in $\mathbb{C}$

Let $M \subset \mathbb{C}$. A path in M is a continuous map

$$\gamma : [a, b] \to \mathbb{C} \quad (a, b \in \mathbb{R} \text{ mit } a \leq b)$$

with $\gamma([a, b]) \subset M$. A path $\gamma : [a, b] \to \mathbb{C}$ is closed if $\gamma(a) = \gamma(b)$.

Examples

- *Line connecting $z, w \in \mathbb{C}$:* $\quad \gamma : [0, 1] \to \mathbb{C}, \gamma(t) = z + t(w - z)$
- *Positively oriented circle:* $\quad \gamma : [0, 2\pi] \to \mathbb{C}, \quad \gamma(t) = a + re^{it}$
- *Given paths $\gamma_j : [a_j, b_j] \to \mathbb{C}$ $(j = 1, 2)$ with $\gamma_1(b_1) = \gamma_2(a_2)$ define $\gamma_1 \wedge \gamma_2 : [a_1, b_1 + (b_2 - a_2)] \to \mathbb{C}$ as the composed path*

$$\gamma_1 \wedge \gamma_2(t) = \begin{cases} \gamma_1(t), & t \in [a_1, b_1] \\ \gamma_2(t + (a_2 - b_1)), & t \in [b_1, b_1 + (b_2 - a_2)] \end{cases}$$

- *For $z_0, \ldots, z_r \in \mathbb{C}$:* $\quad [z_0, z_1] \wedge [z_1, z_2] \wedge \ldots \wedge [z_{r-1}, z_r]$ *(polygon)*
- *For a path $\gamma : [a, b] \to \mathbb{C}$, we define the reversed path*

$$-\gamma : [a, b] \to \mathbb{C}, \ (-\gamma)(t) = \gamma(a + b - t)$$

Connected sets

For $M \subset \mathbb{C}$, a set $V \subset M$ is called open in M if $\exists U \subset \mathbb{C}$ open with $V = M \cap U$.

Definition

- $M \subset \mathbb{C}$ is path connected $:\Leftrightarrow \forall z, w \in M \exists$ path γ in M from z to w.
- $M \subset \mathbb{C}$ is connected $:\Leftrightarrow$ there are no disjoint non-empty open sets V_1, V_2 in M with
$$M = V_1 \cup V_2.$$

Can show: each path connected set in $\mathbb{C}$ is connected, but not vice versa!

Theorem

For an open set $G \subset \mathbb{C}$, the following are equivalent:

(i) *G is connected*

(ii) *G is path connected*

(iii) *Any two points $z, w \in G$ can be joined in G by a polygon.*

Idea for $(i) \Rightarrow (iii)$. Fix $a \in G$. Show that

$U = \{z \in G; z, a$ can be joined by a polygon in $G\}$ and $G \setminus U$ are both open in G.

Continuous functions

Let $f : D \to \mathbb{C}$ be a function on $D \subset \mathbb{C}$ arbitrary. For $a \in \overline{D}, c \in \mathbb{C}$, define

$$\lim_{z \to a} f(z) = c :\Leftrightarrow \lim_{k \to \infty} f(z_k) = c \text{ whenever } D \ni z_k \xrightarrow{k} a$$

We call $f : D \to \mathbb{C}$ continuous at $a \in D \quad \Leftrightarrow \lim_{z \to a} f(z) = f(a)$.

Lemma (Analysis II)

- *$f : D \to \mathbb{C}$ continuous at $a \in D \Leftrightarrow \forall \epsilon > 0 \exists \delta > 0$ with $f(D_\delta(a) \cap D) \subset D_\epsilon(f(a))$*
- *f is continuous on all of $D \Leftrightarrow \overset{-1}{f}(V)$ is open in $D\ \forall V \subset \mathbb{C}$ open*

Lemma (Analysis II)

If $f, g : D \to \mathbb{C}$ are continuous at $a \in D$ and $\lambda \in \mathbb{C}$, then

- *$f + g$, $f \cdot g$, λf are continuous at a*
- *$\frac{f}{g} : \{z \in D; g(z) \neq 0\} \to \mathbb{C}$ is continuous at a if $g(a) \neq 0$*
- *Compositions of continuous functions are continuous.*

Complex differentiable functions

A function $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) is complex differentiable at $a \in U$

$$\exists \quad f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \in \mathbb{C}.$$

Theorem

If $f, g : U \to \mathbb{C}$ are complex differentiable at $a \in U$ then:

- *f is continuous at a.*
- *$f + g, f \cdot g$ are complex differentiable at a and*

  $$(f+g)'(a) = f'(a) + g'(a), \ (fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

  If $g(a) \neq 0$, then $\frac{f}{g}$ is complex differentiable at a and

  $$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

- *If $h : V \to \mathbb{C}$ complex differentiable at $f(a)$, so is $h \circ f : U \to \mathbb{C}$ at a and*

  $$(h \circ f)'(a) = h'(f(a))f'(a).$$

Examples - Nothing new or?

Examples

- $f \equiv c \quad \Rightarrow \quad f'(z) = 0 \;\; \forall z \in \mathbb{C}$
- $f(z) = z \quad \Rightarrow \quad f'(z) = \lim_{h \to 0} \frac{f(z+h)-f(z)}{h} = 1 \;\; \forall z \in \mathbb{C}$
- $f(z) = z^n \; (n \in \mathbb{N}) \quad \Rightarrow \quad f'(z) = nz^{n-1} \;\; \forall z \in \mathbb{C}$
- $f(z) = z^n \; (n \in \mathbb{Z}_{<0}) \quad \Rightarrow \quad f'(z) = nz^{n-1} \;\; \forall z \in \mathbb{C}^*$
- $p(z) = a_n z^n + \ldots + a_1 z + a_0 \quad \Rightarrow \quad p'(z) = na_n z^{n-1} + \ldots + 2a_2 z + a_1 \;\; \forall z \in \mathbb{C}$
- *Rational functions* $\frac{p}{q}$ *(p, q polynomials) are complex differentiable on* $\{q(z) \neq 0\}$

Counterexample: $f(z) = \overline{z}$ as a function on $\mathbb{R}^2$

$$f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (x, -y).$$

is C^∞ in the real sense, but nowhere complex differentiable

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h} = \begin{cases} 1, & h \in \mathbb{R} \\ -1, & h \in i\mathbb{R} \end{cases} .$$

Cauchy-Riemann equations

Complex differentiability is much stronger than real differentiability! Precisely:

For a function $(x, y) \mapsto \big(u(x, y), v(x, y)\big) \in \mathbb{R}^2$ of two real variables, define

$$u_x = \frac{\partial u}{\partial x}, \ u_y = \frac{\partial u}{\partial y} \quad \text{(same for } v\text{)}.$$

Theorem (Cauchy-Riemann)

Let $U \subset \mathbb{C}$ be open, $f = u + iv : U \to \mathbb{C}$ a function, $z_0 = x_0 + iy_0 \in U$. Equivalent are:

- *f is complex differentiable at z_0.*
- *$f = (u, v)$ is totally differentiable in the real sense at (x_0, y_0) and*

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad u_y(x_0, y_0) = -v_x(x_0, y_0) \ \text{(Cauchy-Riemann eq's)}.$$

In this case:

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0), \quad \det\Big(J_{(u,v)}(x_0, y_0)\Big) = |f'(z_0)|^2.$$

How to check complex differentiability?

Corollary

- *$f = u + iv$ complex differentiable at z_0 $\Rightarrow$ u, v partially differentiable with*

$$(*) \quad u_x(z_0) = v_y(z_0), u_y(z_0) = -v_x(z_0)$$

- *If $f = u + iv$ is C^1 near z_0 with $(*)$ $\Rightarrow$ f is complex differentiable at z_0*

Examples

- *$f(z) = |z| = \sqrt{x^2 + y^2)}$ is nowhere complex differentiable, since $v_x = v_y = 0$, but*

$$u_x(x,y) = \frac{x}{(x^2+y^2)^{1/2}} \neq 0 \ (x \neq 0), u_y(x,y) = \frac{y}{(x^2+y^2)^{1/2}} \neq 0 \ (y \neq 0).$$

and $\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$ does not exist.

- *$f(x + iy) = |xy|^{1/2}$ is partially differentiable at $(0,0)$ with*

$$u_x(0,0) = 0 = v_y(0,0), \qquad u_y(0,0) = 0 = -v_x(0,0),$$

not complex differentiable: $\lim_{h\to 0} \frac{f(h,h)-f(0,0)}{h+ih} = \lim_{h\to 0} \frac{1}{1+i}\frac{|h|}{h}$ does not exist.

If f and $\bar{f}$ are holomorphic, then ...

Definition

- A set $G \subset \mathbb{C}$ is called a domain if $G \subset \mathbb{C}$ is open and connected.
- A function $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) is called holomorphic if it is complex differentiable at every point $z \in U$.

Theorem

Let $f : G \to \mathbb{C}$ be holomorphic on a domain $G \subset \mathbb{C}$. Then f is constant under each of the following conditions:

- $f' \equiv 0$,
- $\operatorname{Re} f \equiv \text{const}$,
- $\operatorname{Im} f \equiv \text{const}$,
- $\bar{f}$ $(= \operatorname{Re} f - i \operatorname{Im} f)$ *is holomorphic,*
- $|f|$ *is holomorphic.*

idea: Under any of the first three conditions $u_x = u_y = v_x = v_y \equiv 0$.

Which functions are real parts of holomorphic functions?

Recall that a C^2-function $u : U \to \mathbb{C}$ is said to be harmonic if

$$\Delta u \ (= u_{xx} + u_{yy}) \equiv 0 \text{ on } U.$$

Real and imaginary parts of holomorphic functions are harmonic. Precisely:

Theorem

- *$f : U \to \mathbb{C}$ holomorphic $\Rightarrow u = \mathrm{Re}\, f, v = \mathrm{Im}\, f$ are harmonic on U*
- *$u : D_r(c) \to \mathbb{R}$ harmonic $\Rightarrow \exists$ (unique up to real constants) $v : D_r(c) \to \mathbb{R}$ such that $f = u + iv$ is holomorphic on $D_r(c)$.*

Idea: If $f = u + iv$ is holomorphic, then we shall see later that u, v are C^∞.

$$\Rightarrow \text{(CRDE's and Schwarz lemma) } u_{xx} + u_{yy} = (v_y)_x - (v_x)_y \equiv 0.$$

In the second part the unique solution with $v(0) = 0$ is

$$v(x, y) = \Big(\int_0^1 -u_y(tx, ty)\mathrm{d}t \Big) x + \Big(\int_0^1 u_x(tx, ty)\mathrm{d}t \Big) y.$$

Sequences and series of functions

Let $(a_n)_{n\geq 0}$ be a sequence in $\mathbb{C}$.

Aim: Study functions given by a power series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$.

Definition

Let $f_n : D \to \mathbb{C}$ $(n \in \mathbb{N})$, $f : D \to \mathbb{C}$ be functions. Then $(f_n)_{n\geq 0}$ converges to f on D

- uniformly $:\Leftrightarrow \forall \epsilon \exists n_0 \in \mathbb{N}$ with $|f_n(z) - f(z)| < \epsilon \forall n \geq n_0 \forall z \in D$
- uniformly on compact subsets:$\Leftrightarrow \forall K \subset D$ cpct $(f_n)_{n\geq 0}$ converges to f uniformly on K

Continuity is preserved under uniform convergence on compact subsets.

Definition

Let $f_n : D \to \mathbb{C}$ $(n \in \mathbb{N})$ be functions and $s_N = \sum_{n=0}^{N} f_n \quad (N \in \mathbb{N})$. Define

- $\sum_{n=0}^{\infty} f_n = (s_N)_{N\geq 0}$
- $\sum_{n=0}^{\infty} f_n$ converges pointwise, uniformly (on compact subsets) if $(s_N)_{N\geq 0}$ does

Power series

How to test uniform convergence?

Theorem (W-M Test)

Let $f_n : D \to \mathbb{C}$ $(n \in \mathbb{N})$ be functions, $c_n \in \mathbb{R}$ with

- $|f_n(z)| \leq c_n \; \forall n \in \mathbb{N}$ *and* $\forall z \in D$
- $\sum_{n=0}^{\infty} c_n < \infty$

Then $\sum_{n=0}^{\infty} f_n$ *converges uniformly on* D*.*

Theorem (Power series)

Let $(a_n)_{n\geq 0}$ *be a sequence in* $\mathbb{C}$*,* $a \in \mathbb{C}$ *and* $r > 0$*.*

- $(a_n r^n)_{n\geq 0}$ *bounded* $\Rightarrow \sum_{n=0}^{\infty} a_n(z-a)^n$ *converges absolutely* $\forall z \in D_r(a)$*.*
- $R = \sup\{\rho \geq 0; (a_n\rho^n)_{n\geq 0}$ *bounded*$\}$ $(\in [0, \infty]) \Rightarrow \sum_{n=0}^{\infty} a_n(z-a)^n$ *converges uniformly on all compact subsets of* $D_R(a)$ *and diverges* $\forall z \notin \overline{D}_R(a)$

Radius of convergence

Definition

The unique number $R \in [0, \infty]$ with the property

$$\sum_{n=0}^{\infty} a_n(z-a)^n \text{ converges } \forall z \in D_R(a) \text{ and diverges } \forall z \notin \overline{D}_R(a)$$

is called the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$.

How can one calculate the radius of convergence?

Theorem (Cauchy-Hadamard)

The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ is given by

$$R = 1/\overline{\lim}_{n\to\infty}|a_n|^{1/n} \qquad (c/0 = \infty \text{ and } c/\infty = 0 \text{ for } c > 0).$$

Examples

- $a_n = n^n \quad \Rightarrow \quad R = 1/\overline{\lim}_{n\to\infty}|n^n|^{1/n} = 0$
- $a_n = n^k \quad \Rightarrow \quad R = 1/\overline{\lim}_{n\to\infty}|n^k|^{1/n} = 1$
- $a_n = n^{-n} \quad \Rightarrow \quad R = 1/\overline{\lim}_{n\to\infty}|n^{-n}|^{1/n} = \infty$

Analytic functions

Functions $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ representable by a power series are called analytic.
Analytic functions are complex differentiable, even much better:

Theorem (Termwise differentiation)

If $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ has positive radius of convergence $R \in]0,\infty]$, then

- $f : D_R(a) \to \mathbb{C}$ *is complex differentiable with*

$$f'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1}(z-a)^n \quad \forall z \in D_R(a)$$

- *the differentiated power series has radius of convergence R.*

Corollary

If $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ has positive radius of convergence $R \in]0,\infty]$, then
$f : D_R(a) \to \mathbb{C}$ is infinitely often complex differentiable with

- $f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}\ \forall z \in D_R(a)$
- $a_k = \frac{f^{(k)}(a)}{k!}\ \forall k \in \mathbb{N}.$

Exponential and trigonometric functions

Definition

The complex exponential function, cosine and sine are defined by $\exp, \cos, \sin : \mathbb{C} \to \mathbb{C}$,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Theorem

(i) $\exp, \cos, \sin$ *are holomorphic on* $\mathbb{C}$ *with* $\exp' = \exp, \cos' = -\sin, \sin' = \cos$.

(ii) $\cos z = \frac{\exp(iz)+\exp(-iz)}{2}$, $\sin z = \frac{\exp(iz)-\exp(-iz)}{2i}$, $\exp(iz) = \cos z + i \sin z$ $(z \in \mathbb{C})$.

(iii) $\exp(0) = 1$, $\exp(i\frac{\pi}{2}) = i$, $\exp(i\pi) = -1$, $\exp(i\frac{3}{2}\pi) = -i$, $\exp(2\pi i) = 1$.

Notation: For $z \in \mathbb{C}$ we also write $e^z := \exp(z)$. The unit circle is given by

$$\mathbb{T} := \{z \in \mathbb{C}; |z| = 1\} = \{e^{it};\ t \in \mathbb{R}\} = \{e^{it};\ t \in [\theta_0, \theta_0 + 2\pi[\} \ \ (\theta_0 \in \mathbb{R}).$$

Properties of exp, cos, sin

Theorem (Functional equations)

For $z, w \in \mathbb{C}$, $n \in \mathbb{N}$,

- $e^z e^w = e^{z+w}$, $e^z e^{-z} = e^0 = 1$
- $\sin(z+w) = \sin z \cos w + \cos z \sin w, \quad \cos(z+w) = \cos z \cos w - \sin z \sin w$
- $(\cos z + i \sin z)^n = \cos(nz) + i \sin(nz)$
- $\sin^2 z + \cos^2 z = 1.$

Theorem (Typical values)

For $z \in \mathbb{C}$,

- $|e^z| = e^{\mathrm{Re}\, z}, \quad \overline{e^z} = e^{\overline{z}}$
- $e^z = 1 \;\Leftrightarrow\; z \in 2\pi i\mathbb{Z} \;\; (= \{2\pi i k; k \in \mathbb{Z}\})$
- $\sin z = 0 \;\Leftrightarrow\; z \in \pi\mathbb{Z}$
- $\cos z = 0 \;\Leftrightarrow\; z \in \frac{\pi}{2} + \pi\mathbb{Z}.$

Branches of the logarithm

For $z \in \mathbb{C}^*$, $\theta_0 \in \mathbb{R}$ and $u \in \mathbb{C}$, we have $e^u = z \Leftrightarrow u$ is of the form

$$u = \log|z| + i(\arg_{\theta_0}(z) + 2\pi k) \quad (k \in \mathbb{Z}).$$

Definition

Let $G \subset \mathbb{C}$ be a domain. A branch of the logarithm (Zweig) on G is a continuous function $f : G \to \mathbb{C}$ with

$$\exp(f(z)) = z \quad \text{for all } z \in G.$$

Each branch of the logarithm $f : G \to \mathbb{C}$ is holomorphic and satisfies $\forall z \in G$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \left(\frac{e^{f(z+h)} - e^{f(z)}}{f(z+h) - f(z)} \right)^{-1} = \frac{1}{e^{f(z)}} = \frac{1}{z}.$$

Standard complex logarithms

For $\theta \in \mathbb{R}$ define

$$\mathbb{C}_\theta = \{z \in \mathbb{C}^*;\ \arg_\theta(z) \in]\theta, \theta + 2\pi[\} = \mathbb{C} \setminus \{re^{i\theta};\ r \in [0, \infty)\} \quad (\theta \in \mathbb{R}).$$

Theorem (k-th branch of the logarithm on $\mathbb{C}_\theta$)

For $\theta \in \mathbb{R}$ and $k \in \mathbb{Z}$

$$\log_{\theta,k} : \mathbb{C}_\theta \to \mathbb{C},\ z \mapsto \log|z| + i(\arg_\theta + 2\pi k)$$

is holomorphic such that $\forall z \in \mathbb{C}_\theta$

- $e^{\log_{\theta,k}(z)} = z$
- $\frac{\mathrm{d}}{\mathrm{d}z}\Big(\log_{\theta,k}(z)\Big) = \frac{1}{z}$

The mapping $\log_{\theta,k} : \mathbb{C}_\theta \to \mathbb{R} \times]\theta + 2\pi k, \theta + 2\pi(k+1)[$ is bijective.

Definition

The principal branch of the logarithm (Hauptzweig) is defined as

$$\log = \log_{-\pi,0} : \mathbb{C}_{-\pi} = \mathbb{C} \setminus (-\infty, 0] \ \to\ \mathbb{C},\ z \mapsto \log|z| + i\arg_{-\pi}(z)$$

Powers and roots

As in the real case exp and log can be used to define powers of complex numbers with complex exponents.

Theorem

Given $\theta \in \mathbb{R}$, $k \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$,

$$\mathbb{C}_\theta \to \mathbb{C},\; z \mapsto z^{(\alpha,\theta,k)} \;=\; \exp\left(\alpha \log_{\theta,k}(z)\right) \;=\; |z|^\alpha e^{i\alpha(\arg_\theta(z)+2\pi k)}$$

defines a holomorphic function such that for $\alpha, \beta \in \mathbb{C}$ und $z \in \mathbb{C}_\theta$,

- $z^{(\alpha,\theta,k)}\, z^{(\beta,\theta,k)} = z^{(\alpha+\beta,\theta,k)}$,
- $\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{(\alpha,\theta,k)}\right) = \alpha z^{(\alpha-1,\theta,k)}$.

Definition (Principal branch of the complex powers)

For $z \in \mathbb{C}_{-\pi} = \mathbb{C} \setminus (-\infty, 0]$ and $\alpha \in \mathbb{C}$, we simply write

$$z^\alpha \;=\; z^{(\alpha,-\pi,0)} \;=\; e^{\alpha(\log|z|+i\arg_{-\pi}(z))} \;=\; |z|^\alpha\; e^{i\alpha\arg_{-\pi}(z)}.$$

n^{th} roots

Lemma (n-th roots)

For $w \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$, the equation

$$z^n = w$$

has exactly n distinct solutions. If $w \in \mathbb{C}_\theta$, then the n solutions are given by

$$z_k = w^{(\frac{1}{n},\theta,k)} \quad (k = 0, \dots, n-1).$$

For $w = 1$ one obtains the n-th roots of unity (Einheitswurzeln):

Corollary (Exercise 1)

The n distinct solutions of the equation $z^n = 1$ are given by

$$z_k = e^{i\frac{2\pi k}{n}} \qquad (k = 0, \dots, n-1)$$

$\mathbb{C}$-valued Riemann integrals

A function $f : [a, b] \to \mathbb{C}$ is called

- Riemann integrable if $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are Riemann integrable. In this case
$$\int_a^b f \mathrm{d}t := \int_a^b \mathrm{Re}\, f \mathrm{d}t + i \int_a^b \mathrm{Im}\, f \mathrm{d}t.$$
pause
- continuously differentiable if $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are.
- piecewise continuously differentiable $:\Leftrightarrow \exists$ partition $a = t_0 < t_1 < \ldots < t_n = b$ such that $f|_{[t_{i-1}, t_i]}$ is continuously differentiable $\forall i = 1, \ldots, n$.

Lemma

Let $I = [a, b]$ be a compact interval and $f : I \to \mathbb{C}$ a function.

- $\mathrm{RI}(I, \mathbb{C}) \to \mathbb{C}$, $f \mapsto \int_a^b f \mathrm{d}t$ *is* $\mathbb{C}$*-linear*
- f *and* $|f|$ *Riemann integrable* $\Rightarrow \left| \int_a^b f \mathrm{d}t \right| \leq \int_a^b |f| \mathrm{d}t$
- f *continuously differentiable* $\Rightarrow \int_a^b f' \mathrm{d}t = f(b) - f(a)$.

Contour integrals

Definition

- A contour (Integrationsweg) in M is a piecewise continuously differentiable function $\gamma : [a, b] \to \mathbb{C}$ with $\gamma([a, b]) \subset M$. We call γ closed if $\gamma(a) = \gamma(b)$.
- If γ is a contour with $\gamma_{[t_{i-1} - t_i]}$ continuously differentiable $\forall i$, we set

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f \circ \gamma(t)\, \gamma'(t)\mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f \circ \gamma(t)\, \gamma'(t)\mathrm{d}t.$$

The definition does not depend on the choice of the partition $a = t_0 < \ldots < t_n = b$.
The set $\mathrm{Sp}(\gamma) = \gamma([a, b])$ is called the trace (Spur) of γ.

Lemma

Let $\gamma, \gamma_1, \gamma_2$ be contours in $M \subset \mathbb{C}$ and let $f, g : M \to \mathbb{C}$ be continuous. Then:

- $\int_\gamma (\alpha f + \beta g)\mathrm{d}z = \alpha \int_\gamma f\mathrm{d}z + \beta \int_\gamma g\mathrm{d}z$
- *endpoint of* γ_1 = *starting point of* $\gamma_2 \;\Rightarrow\; \int_{\gamma_1 \wedge \gamma_2} f\mathrm{d}z = \int_{\gamma_1} f\mathrm{d}z + \int_{\gamma_2} f\mathrm{d}z$

Parameter transformations

Using the chain rule one obtains:

Theorem

Let $\gamma : [a, b] \to \mathbb{C}$ *be a contour,* $f : \mathrm{Sp}(\gamma) \to \mathbb{C}$ *continuous.*

- $\int_{-\gamma} f \mathrm{d}z = -\int_\gamma f \mathrm{d}z$
- $\varphi : [c, d] \to [a, b]$ C^1*-function with* $\varphi'(t) > 0$ *for* $t \in [c, d]$ *and* $\varphi(c) = a$, $\varphi(d) = b$ $\Rightarrow \tilde{\gamma} = \gamma \circ \varphi : [c, d] \to \mathbb{C}$ *is a contour with*

$$\int_{\tilde{\gamma}} f \mathrm{d}z = \int_\gamma f \mathrm{d}z.$$

Definition

The length of a contour $\gamma : [a, b] \to \mathbb{C}$ is defined as

$$L(\gamma) = \int_a^b |\gamma'(t)| \mathrm{d}t \left(:= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |\gamma'(t)| \mathrm{d}t, \text{ if } \gamma_{[t_{j-1}, t_j]} \text{ are } C^1 \forall j \right).$$

The length of contours is preserved when replacing γ by $-\gamma$ or by $\gamma \circ \varphi$ with a C^1-invertible parameter transformation φ with $\varphi' > 0$.

Fundamental estimate for contours

If $f : D \to \mathbb{C}$ is a function, then for $M \subset D$, we define

$$\|f\|_M = \sup_{z \in M} |f(z)| \quad \Big(\in [0, \infty] \Big).$$

Theorem (Fundamental estimate)

If $\gamma : [a, b] \to \mathbb{C}$ is a contour and $f : \mathrm{Sp}(\gamma) \to \mathbb{C}$ is continuous, then

$$\Big| \int_\gamma f \mathrm{d}z \Big| \leq L(\gamma) \, \|f\|_{\mathrm{Sp}(\gamma)}.$$

As a consequence one can exchange limits and path integrals provided the integrands converge uniformly.

Corollary

Let $\gamma : [a, b] \to \mathbb{C}$ be a contour and $f_n, f : \mathrm{Sp}(\gamma) \to \mathbb{C}$ continuous functions such that $(f_n)_{n \geq 0}$ converges uniformly on $\mathrm{Sp}(\gamma)$ to f. Then

$$\lim_{n \to \infty} \int_\gamma f_n \mathrm{d}z = \int_\gamma f \mathrm{d}z.$$

Fundamental theorem for contour integrals

We write $\int_{\partial D_r(a)} f\mathrm{d}z = \int_\gamma f\mathrm{d}z$ with $\gamma(t) = a + re^{it}$ $(t \in [0, 2\pi])$.

Examples

For $a \in \mathbb{C}$, $r > 0$ and all $n \in \mathbb{Z}$

$$\int_{\partial D_r(a)} (z-a)^n \mathrm{d}z = \int_0^{2\pi} (re^{it})^n rie^{it} \mathrm{d}t = \int_0^{2\pi} i\, r^{n+1}\, e^{i(n+1)t} \mathrm{d}t$$

$$= \begin{cases} 0\ , & n \neq -1 \\ 2\pi i, & n = -1 \end{cases} .$$

Theorem (Fundamental theorem for contour integrals)

If $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) is continuous and $F : U \to \mathbb{C}$ holomorphic with $F' = f$, then

$$\int_\gamma f\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)) \quad \forall \text{ contours } \gamma : [a, b] \to U.$$

Existence of primitives

Definition

Let $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) be continuous. A function $F : U \to \mathbb{C}$ is called a primitive (Stammfunktion) for f if F is holomorphic with $F' = f$.

Examples

- If $f : U \to \mathbb{C}$ has a primitive and γ is a closed contour in U, then $\int_\gamma f \mathrm{d}z = 0$.
- In particular: $\int_{\partial D_r(a)} (z-a)^n \mathrm{d}z = 0$ for all $n \in \mathbb{Z}$ with $n \neq -1$.
- $\int_{\partial D_r(a)} \frac{1}{z-a} \mathrm{d}z = 2\pi i \neq 0 \quad \Rightarrow \quad \frac{1}{z-a}$ has no primitive on $\mathbb{C} \setminus \{a\}$.

Theorem (6.5)

Let $G \subset \mathbb{C}$ be a domain and $f : G \to \mathbb{C}$ continuous. Then equivalent are:

(i) *f has a primitive on G.*

(ii) *$\int_\gamma f \mathrm{d}z = 0$ for each closed contour γ in G.*

Idea for $(ii) \Rightarrow (i)$: For fixed $a \in G$ define a primitive of f by

$$F(z) = \int_\gamma f(\xi) \mathrm{d}\xi, \text{ if } \gamma \text{ is a contour in } G \text{ with } A(\gamma) = a, E(\gamma) = z,$$

Goursat's lemma

Theorem (6.6)

If $G \subset \mathbb{C}$ is open and convex, then (i) and (ii) are equivalent to

(iii) $\int_\delta f\mathrm{d}z = 0$ *for each triangle* $\delta = [a, b] \wedge [b, c] \wedge [c, a]$ *in* G.

For $a, b, c \in \mathbb{C}$ we denote the smallest convex set $\Delta \ni a, b, c$ (closed triangle) by

$$\Delta = \Delta(a, b, c) = \{t_1 a + t_2 b + t_3 c;\ 0 \leq t_1, t_2, t_3 \leq 1 \text{ und } t_1 + t_2 + t_3 = 1\} \subset \mathbb{C}$$

and write $\int_{\partial\Delta} f\, d\, z = \int_{[a,b]\wedge[b,c]\wedge[c,a]} f\, d\, z$.

Theorem (Goursat's lemma)

Let $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) be holomorphic, $\Delta = \Delta(a, b, c) \subset U$ a closed triangle

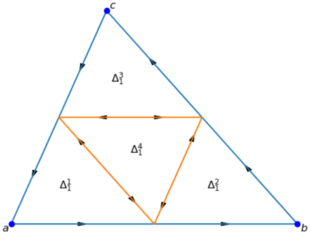
$$\Rightarrow \quad \int_{\partial\Delta} f\mathrm{d}z = 0.$$

Idea: Subdivide Δ by connecting the midpoints of the edges

$$|\underbrace{\int_{\partial\Delta} f\mathrm{d}z}_{=I}| = |\sum_{i=1}^{4} \int_{\partial\Delta_1^i} f\mathrm{d}z| \leq 4 \max_{1\leq i\leq 4} \left| \int_{\partial\Delta_1^i} f\mathrm{d}z \right|$$

Goursat's lemma

and continue like this forever:



One obtains the estimates (with $\{z_0\} = \bigcap_{n\geq 1} \Delta_n$)

$$|I| \leq 4^n \Big| \underbrace{\int_{\partial\Delta_n} f(z_0) + (z - z_0)f'(z_0)\mathrm{d}z}_{=0} + \int_{\partial\Delta_n} (z - z_0)r(z)\mathrm{d}z \Big|$$

$$\leq 4^n \left(\frac{1}{4}\right)^n L(\partial\Delta)^2 \sup_{z\in\overline{D}_{\mathrm{diam}(\Delta_n)}(z_0)} |r(z)| \xrightarrow{n} 0$$

Man benutzt dabei, dass $\mathrm{diam}(\Delta_n) \leq L(\partial\Delta_n) = \left(\frac{1}{2}\right)^n L(\partial\Delta)$.

Cauchy's integral theorem and formula

Corollary (Cauchy's integral thm for convex domains)

Let $G \subset \mathbb{C}$ be a convex domain and let $f : G \to \mathbb{C}$ be holomorphic

$$\Rightarrow \quad \int_\gamma f \mathrm{d}z = 0 \quad \textit{for each closed contour } \gamma \textit{ in } G.$$

Corollary (Cauchy's integral thm for circles)

Let $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) be holomorphic and let $\overline{D}_r(a) \subset U$ be a closed disc.

$$\Rightarrow \quad \int_{\partial D_r(a)} f \mathrm{d}z = 0.$$

Theorem (Cauchy's integral formula for circles)

Let $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) be holomorphic (+ f' continuous) and let $\overline{D}_r(a) \subset U$

$$\Rightarrow \quad f(z) = \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi \qquad \textit{for all } z \in D_r(a).$$

Taylor's formula

Theorem (6.13)

Let $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) be holomorphic (+ f' continuous) and let $D_R(a) \subset U$. Then f is infinitely often complex differentiable and

- $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$ *for all* $z \in D_R(a)$,
- $f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\partial D_r(a)} \frac{f(\xi)}{(\xi - a)^{n+1}} \, \mathrm{d}\xi$ *for* $0 < r < R$ *and* $n \in \mathbb{N}$.

Corollary (6.14)

$f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) holomorphic $\Rightarrow$ f infinitely often complex differentiable. In particular f' is automatically continuous.

Proof. May suppose that U is convex.

$$\Rightarrow \quad \text{(Goursat)} \int_{\partial\Delta} f \, dz = 0 \; \forall \text{ closed triangles } \Delta \subset U$$

Thus f has a primitive F. But then F and $f = F'$ are infinitely often complex differentiable.

CIF for derivatives and Taylor expansion

Theorem (Cauchy's integral formula for derivatives)

Let $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) be holomorphic and $\overline{D}_r(a) \subset U$

$$\Rightarrow \quad f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial D_r(a)} \frac{f(\xi)}{(\xi - w)^{n+1}} \, \mathrm{d}\xi \quad \forall n \geq 0 \; \forall w \in D_r(a)$$

Corollary (6.16)

Let $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) be holomorphic and let $a \in U$.
Then f admits a representation as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n \quad \forall z \in D_r(a)$$

on each disc $D_r(a) \subset U$ with uniquely determined coefficients

$$a_n = \frac{f^{(n)}(a)}{n!} \quad \forall \; n \in \mathbb{N}.$$

Theorems of Morera and Weierstraß

Theorem (Morera's theorem)

Let $U \subset \mathbb{C}$ open and let $f : U \to \mathbb{C}$ be continuous with

$$\int_{\partial\Delta} f \mathrm{d}z = 0 \quad \text{for each closed triangle } \Delta \subset U.$$

Then f is holomorphic.

Idea: Theorem 6.6 $\Rightarrow$ f has a primitive F on each disc $D_r(a) \subset U$

$$\Rightarrow f = F' \text{ is infinitely often complex differentiable}$$

Theorem (Weierstraß' theorem)

If $(f_n) \xrightarrow{n} f$ uniformly on all compact subsets of $U \subset \mathbb{C}$ open and all f_n are holomorphic, then f is holomorphic and $\forall k \in \mathbb{N}$

$$(f_n^{(k)}) \xrightarrow{n} f^{(k)} \quad \text{uniformly on all compact subsets of } U.$$

Morera's theorem $\Rightarrow$ f holomorphic. The rest follows with CIF for the derivatives.

Liouville

Wir schreiben: $\mathcal{O}(U) = \{f;\ f:\ U \to \mathbb{C} \text{ ist holomorph}\}$.

Die Funktionen $f \in \mathcal{O}(\mathbb{C})$ heißen ganze Funktionen.

Theorem (Satz von Liouville)

Jede beschränkte ganze Funktion $f \in \mathcal{O}(\mathbb{C})$ ist konstant.

Idee: Sei $M = \|f\|_{\mathbb{C}}$. Dann gilt für alle $r > 0$ und $n \geq 1$

$$\left|\frac{f^{(n)}(0)}{n!}\right| = \left|\frac{1}{2\pi i}\int_{\partial D_r(0)} \frac{f(\xi)}{\xi^{n+1}}\,\mathrm{d}\xi\right| \leq \frac{L(\partial D_r(0))}{2\pi}\,\frac{M}{r^{n+1}} = \frac{M}{r^n} \xrightarrow{(r\to\infty)} 0$$

Also ist $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \equiv f(0)$.

Theorem (Cauchysche Ungleichungen)

Sei $f \in \mathcal{O}(D_R(a))$. Dann gilt für $0 < r < R$

$$|f^{(n)}(a)| \leq \frac{n!}{r^n}\|f\|_{\partial D_r(a)} \quad (n \in \mathbb{N}).$$

Folgerung: $f \in \mathcal{O}(\mathbb{C})$ mit $|f(z)| \leq c|z|^N$ für $|z| > R \Rightarrow f$ Polynom mit $\deg f \leq N$.

Vielfachheit von Nullstellen

Lemma (7.5)

Sei $f \in \mathcal{O}(U)$, $a \in U$ und f in keiner Umgebung von a identisch $0 \Rightarrow \exists! N \in \mathbb{N}$ mit

$$\exists g \in \mathcal{O}(U) \text{ mit } g(a) \neq 0 \text{ und } f(z) = (z-a)^N g(z) \quad \forall z \in U.$$

Man nennt

- N die Vielfachheit der Nullstelle a von f
- a Nullstelle unendlicher Vielfachheit, wenn $f^{(n)}(a) = 0 \forall n \in \mathbb{N}$.

Theorem (Fundamentalsatz der Algebra)

Sei $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \in \mathbb{C}[z]$ mit $n \geq 1$ und $a_n \neq 0$. Dann hat

- *p mindestens eine Nullstelle in $\mathbb{C}$ und ist von der Form*
- *$p(z) = a_n \prod_{i=1}^n (z_i - c_i)$ mit geeigneten $c_1, \ldots, c_n \in \mathbb{C}$.*

Idee: Hätte p keine Nullstelle in $\mathbb{C}$, so wäre $1/p$ wegen

$$\lim_{|z| \to \infty} |p(z)| = \infty$$

eine beschränkte ganze Funktion und daher $1/p \equiv 0$.

Riemannscher Hebbarkeitssatz und Identitätssatz

Theorem (Riemannscher Hebbarkeitssatz)

Ist $f \in \mathcal{O}(U \setminus \{a\})$ ($a \in U \subset \mathbb{C}$ offen) beschränkt auf $D_r(a) \setminus \{a\} \subset U$ für ein $r > 0$, dann

$$\exists\, g \in \mathcal{O}(U) \text{ mit } f = g|_{U \setminus \{a\}}.$$

Idee: Zeige, dass

$$F : U \to \mathbb{C}, \quad F(z) = \begin{cases} (z-a)^2 f(z), & z \neq a \\ 0 \qquad\qquad , & z = a \end{cases}$$

holomorph ist mit $F(a) = F'(a) = 0 \Rightarrow$ (7.5) $F \in (z-a)^2 \mathcal{O}(U)$.

Theorem (Identitätssatz)

Sei $G \subset \mathbb{C}$ ein Gebiet, $A \subset G$ nicht diskret (d.h. A besitze einen Häufungspkt in G). Dann gilt

$$f, g \in \mathcal{O}(G) \text{ mit } f = g \text{ auf } A \quad \Rightarrow \quad f = g \quad \text{auf ganz } G.$$

Beispiel: Sind $f, g \in \mathcal{O}(D_1(0))$ mit $f(1/n) = g(1/n)$ für fast alle n, so ist $f = g$.

Maximumprinzip

Theorem (Maximumprinzip)

Sei $G \subset \mathbb{C}$ ein Gebiet und $f \in \mathcal{O}(G)$, $g \in C(\overline{G})$ mit $g|_G \in \mathcal{O}(G)$.

- *Hat $|f|$ in einem $a \in G$ ein lokales Maximum, so ist $f \equiv const$.*
- *Ist G beschränkt und $g|_G$ nicht konstant, so gilt $|g(z)| < \|g\|_{\partial G} \forall z \in G$.*

Idee: Sei $|f| \leq |f(a)|$ auf $D_R(a) \subset G \Rightarrow \forall 0 < r < R$

$$0 \overset{CIF}{=} \left| f(a) - \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{f(\xi)}{\xi - a} \mathrm{d}\xi \right| = \left| f(a) - \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) \mathrm{d}t \right|$$

$$\geq |f(a)| - \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})| \mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{|f(a)| - |f(a + re^{it})|}_{\geq 0 \text{ stetig in } t} \mathrm{d}t \geq 0$$

$\Rightarrow |f| \equiv |f(a)|$ auf $D_R(a) \overset{2.19}{\Rightarrow} f \equiv f(a)$ auf $D_R(a)$

$\Rightarrow$ (Identitätssatz) $f \equiv f(a)$ auf G

Minimumprinzip und Gebietstreue

Theorem (Minimumprinzip = 7.13)

Sei $G \subset \mathbb{C}$ ein Gebiet und $f \in \mathcal{O}(G)$, $g \in C(\overline{G})$ mit $g|_G \in \mathcal{O}(G)$.

- *Hat $|f|$ in einem $a \in G$ ein lokales Minimum, so ist $f(a) = 0$ oder $f \equiv const$.*
- *Ist G beschränkt und $g|_G$ nicht konstant, so hat g eine Nullstelle in G oder*

$$|g(z)| > \min\{|g(w)|;\ w \in \partial G\}\ \forall\, z \in G.$$

Theorem (Satz von der Gebietstreue = 7.14)

Sei $G \subset \mathbb{C}$ ein Gebiet und $f \in \mathcal{O}(G)$ nicht konstant $\Rightarrow$ $f(G)$ ist ein Gebiet und

$$f \text{ ist offen, d.h. } f(U) \subset \mathbb{C} \text{ ist offen } \forall U \subset G \text{ offen.}$$

Connected components

Let $U \subset \mathbb{C}$ be open. An equivalence relation on U is defined by

$$z \sim w \;\Leftrightarrow\; \text{there is a path } \gamma \text{ from } z \text{ to } w \text{ in } U.$$

The equivalence classes are called path components of U or simply components of U.

Lemma (8.2)

Let $U \subset \mathbb{C}$ be open and $z \in U$.

(a) *The equivalence class $C(z)$ of z in U is open and connected.*

(b) *U is the disjoint union of its components.*

(c) *$M \subset U$ connected or path connected with $M \cap C(z) \neq \emptyset \;\Rightarrow\; M \subset C(z)$.*

Examples (8.3)

$K \subset \mathbb{C}$ compact $\;\Rightarrow\; \exists!$ unbounded component C_∞ in $\mathbb{C} \setminus K$. We have

$$\{z \in \mathbb{C};\ |z| > \sup_{w \in K} |w|\} \subset C_\infty.$$

A Cauchy-type integral

Lemma (8.4)

Let γ be a contour in $\mathbb{C}$, $\varphi, \psi : \mathrm{Sp}(\gamma) \to \mathbb{C}$ continuous and $U = \mathbb{C} \setminus \varphi(\mathrm{Sp}(\gamma))$. Then

$$f : U \to \mathbb{C}, \quad f(z) = \int_\gamma \frac{\psi(\xi)}{\varphi(\xi) - z} \mathrm{d}\xi$$

is holomorphic with $\lim_{|z|\to\infty} f(z) = 0$. For all $z \in U$ and $n \in \mathbb{N}$,

$$f^{(n)}(z) = n! \int_\gamma \frac{\psi(\xi)}{(\varphi(\xi) - z)^{n+1}} \mathrm{d}\xi.$$

Idee: For $z \in D_r(a) \subset U = \mathbb{C} \setminus \varphi(\mathrm{Sp}(\gamma)$, the series

$$\frac{\psi(\xi)}{\varphi(\xi) - z} = \frac{\psi(\xi)}{(\varphi(\xi) - a)(1 - \frac{z-a}{\varphi(\xi)-a})} = \sum_{n=0}^{\infty} \frac{\psi(\xi)}{(\varphi(\xi) - a)^{n+1}} (z-a)^n$$

converges uniformly for $\xi \in \mathrm{Sp}(\gamma)$ by the WM-test. Hence

$$f(z) = \sum_{n=0}^{\infty} \Big(\int_\gamma \frac{\psi(\xi)}{(\varphi(\xi) - a)^{n+1}} \mathrm{d}\xi \Big) (z-a)^n.$$

Winding number

Theorem (8.5)

Let γ be a closed contour in $\mathbb{C}$. Then

$$\operatorname{ind}_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z} \mathrm{d}\xi \in \mathbb{Z} \ \forall z \in \mathbb{C} \setminus \operatorname{Sp}(\gamma)$$

is constant on each component and vanishes on the unbounded component.

Idee: Show that $\exp\left(\int_\gamma \frac{\mathrm{d}\xi}{\xi - z}\right) = 1$ for all $z \in \mathbb{C} \setminus \operatorname{Sp}(\gamma)$.

Definition (8.6)

The winding number (Umlaufzahl) or index of γ relative to z is defined as the integer

$$\operatorname{ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z} \mathrm{d}\xi.$$

Examples (8.7)

For $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = a + re^{imt}$ $(a \in \mathbb{C}, m \in \mathbb{Z})$,

$$\operatorname{ind}_\gamma(z) = \begin{cases} m, & z \in D_r(a), \\ 0, & z \notin \overline{D_r}(a). \end{cases}$$

Integrals along cycles

A cycle (Zyklus) in $M \subset \mathbb{C}$ is a tuple $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of closed contours in M.
For cycles $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Delta = (\delta_1, \ldots, \delta_m)$ in $\mathbb{C}$, we define

- $\mathrm{Sp}(\Gamma) = \mathrm{Sp}(\gamma_1) \cup \ldots \cup \mathrm{Sp}(\gamma_n)$
- $-\Gamma = (-\gamma_1, \ldots, -\gamma_n)$
- $\Gamma + \Delta = (\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m)$
- $\int_\Gamma f \mathrm{d}z = \sum_{i=1}^n \int_{\gamma_i} f \mathrm{d}z$ if $f : \mathrm{Sp}(\Gamma) \to \mathbb{C}$ is continuous
- $\mathrm{ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}\xi}{\xi - z} = \sum_{i=1}^n \mathrm{ind}_{\gamma_i}(z) \quad (z \in \mathbb{C} \setminus \mathrm{Sp}(\Gamma))$

Definition

Let Γ be a cycle or a single contour in $\mathbb{C}$. We define the interior of Γ as the set

$$\mathrm{Int}(\Gamma) = \{z \in \mathbb{C} \setminus \mathrm{Sp}(\Gamma); \mathrm{ind}_\Gamma(z) \neq 0\}.$$

Since $\mathrm{ind}_\Gamma(\cdot) : \mathbb{C} \setminus \mathrm{Sp}(\Gamma) \to \mathbb{C}$ is continuous, $\mathrm{Int}(\Gamma) \subset \mathbb{C}$ is open.
We identify a single contour γ with the cycle $\Gamma = (\gamma)$.

Global Cauchy theorem

Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ be a cycle in $\mathbb{C}$. Then

- $\mathrm{Int}(\Gamma) \subset \{z \in \mathbb{C};\ |z| \leq \sup_{w \in \mathrm{Sp}(\Gamma)} |w|\}$ and $\mathrm{Int}(\Gamma) \cup \mathrm{Sp}(\Gamma)$ is compact
- $\mathrm{ind}_\Gamma(\cdot)$ is constant on each component of $\mathbb{C} \setminus \mathrm{Sp}(\Gamma)$ and 0 on C_∞.

Theorem (The global Cauchy theorem = 8.10)

Let $f \in \mathcal{O}(U)$ ($U \subset \mathbb{C}$ open). Let Γ be a cycle in U with $\mathrm{Int}(\Gamma) \subset U$. Then

(a) $\int_\Gamma f \mathrm{d}z = 0$,

(b) $\mathrm{ind}_\Gamma(z)\, f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi \quad \forall z \in U \setminus \mathrm{Sp}(\Gamma)$,

(c) $\mathrm{ind}_\Gamma(z)\, f^{(k)}(z) = \frac{k!}{2\pi i} \int_\Gamma \frac{f(\xi)}{(\xi - z)^{k+1}} \mathrm{d}\xi \quad \forall z \in U \setminus \mathrm{Sp}(\Gamma)$ *and* $k \in \mathbb{N}$.

(d) *If Γ_1 and Γ_2 are cycles in U with*

$$\mathrm{ind}_{\Gamma_1}(z) = \mathrm{ind}_{\Gamma_2}(z) \quad \forall z \in \mathbb{C} \setminus U,$$

then

$$\int_{\Gamma_1} f \mathrm{d}z = \int_{\Gamma_2} f \mathrm{d}z.$$

Global Cauchy thm: An application

Examples (8.11)

Let $U = D_t(a) \cap \overline{D}_s(a)^c$ for $0 \leq s < t \leq \infty$ and $f \in \mathcal{O}(U)$.
For $s < r < t$, define $\gamma_r : [0, 2\pi] \to \mathbb{C}, t \mapsto a + re^{it}$. Then for $s < r_1 < r_2 < t$

$$\mathrm{ind}_{\gamma_{r_1}}(z) = \mathrm{ind}_{\gamma_{r_2}}(z) \text{ for all } z \in \mathbb{C} \setminus U$$

and

$$\mathrm{ind}_{(\gamma_{r_2}, -\gamma_{r_1})}(z) = \underbrace{\mathrm{ind}_{\gamma_{r_2}}(z)}_{=1} - \underbrace{\mathrm{ind}_{\gamma_{r_1}}(z)}_{=0} = 1 \ \text{ for all } r_1 < |z| < r_2.$$

Therefore:

1. $\int_{\gamma_r} f \mathrm{d}z$ does not depend on $r \in (s, t)$
2. $f(z) = \frac{1}{2\pi i}\left(\int_{\gamma_{r_2}} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi - \int_{\gamma_{r_1}} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi \right)$ for all $r_1 < |z| < r_2$
3. The integrals in (2) define analytic functions $f_0 \in \mathcal{O}(D_t(a))$, $f_\infty \in \mathcal{O}(\overline{D}_s(a)^c)$ with

$$f = f_0 - f_\infty \text{ on } U.$$

Simply connected domains

Definition (8.12)

- A domain $G \subset \mathbb{C}$ is simply connected if $\mathrm{Int}(\gamma) \subset G$ $\forall$ closed contours γ in G.
- A holomorphic logarithm of $f \in \mathcal{O}(U)$ is a function $g \in \mathcal{O}(U)$ with $e^g = f$.

Theorem (8.13)

Let $G \subset \mathbb{C}$ be a domain. Equivalent are:

(i) *G is simply connected.*

(ii) $\int_\gamma f \mathrm{d}z = 0$ $\forall$ *closed contours γ in G and $f \in \mathcal{O}(G)$.*

(iii) *Each function $f \in \mathcal{O}(G)$ has a primitive.*

(iv) *Each function $f \in \mathcal{O}(G)$ with $0 \notin f(G)$ has a holomorphic logarithm.*

Some ideas: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). Global Cauchy thm and Theorem 6.5.
(iii) $\Rightarrow$ (iv). Aufgabe 20
(iv) $\Rightarrow$ (i). If $1/(z-a) = e^g$, then $1/(z-a) = -(1/z-a)'/1/(z-a) = -g'$.

Criteria for simple connectedness

Theorem (8.14)

Let $G \subset \mathbb{C}$ be a domain.

- *G not simply connected $\Rightarrow$ $\exists\, \emptyset \neq A_1$ compact and A_2 closed with*

$$\mathbb{C} \setminus G = A_1 \cup A_2, \qquad A_1 \cap A_2 = \emptyset.$$

- *$\mathbb{C} \setminus G$ unbounded and connected $\Rightarrow$ G is simply connected.*

Examples (8.15)

- Convex domains are simply connected by Cauchy's integral theorem for convex domains.
- $\mathbb{C}_{-\pi} = \mathbb{C} \setminus (-\infty, 0]$ is simply connected by Theorem 8.14 (but not convex).
- $G = \{z \in \mathbb{C};\ |\mathrm{Im}\, z| < 1\}$ is simply connected (but $\mathbb{C} \setminus G$ is not connected).

Isolated singularities: Definition

Definition (9.1)

An isolated singularity for $f \in \mathcal{O}(U)$ is a point $a \in \mathbb{C} \setminus U$ with $\dot{D}_r(a) \subset U$ for some $r > 0$.

An isolated singularity a for f is called

- removable if $\exists\, g \in \mathcal{O}(U \cup \{a\}$ with $f = g|_U$,
- pole if $\lim_{z \to a} |f(z)| = \infty$,
- essential if a is neither removable nor a pole.

Examples (9.2)

0 is an isolated singularity for the functions

$$f(z) = \frac{\sin z}{z} \text{ (removable)}, \quad g(z) = \frac{1}{z^n} (n \in \mathbb{N}^*) \text{ (pole)}, \quad h(z) = e^{\frac{1}{z}} \text{ (essential)}.$$

If $A \subset U$ is discrete and $f \in \mathcal{O}(U \setminus A)$, then A consists of isolated singularities for f.

Isolated singularities: Characterizations

Theorem (9.4)

For $f \in \mathcal{O}(U \setminus \{a\})$ ($a \in U$), exactly one of the following cases holds:

(i) *a is removable for f,*

(ii) $\exists m \in \mathbb{N}^*, c_1, \ldots, c_m \in \mathbb{C}$ *with* $c_m \neq 0$ *such that a is removable for*

$$U \setminus \{a\} \to \mathbb{C},\ z \mapsto f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k},$$

(iii) $\forall w \in \mathbb{C}\ \exists (z_n)_{n\geq 0}$ *in* $U \setminus \{a\}$ *with* $\lim_{n\to\infty} z_n = a$ *and* $\lim_{n\to\infty} f(z_n) = w$.

f possesses a pole in a in case (ii) and an essential singularity in case (iii).

In case (ii) the numbers $m \in \mathbb{N}^*, c_1, \ldots, c_m$ are uniquely determined.

Definition (9.6)

In the setting of case (ii) one calls

- m the order of the pole a,
- $\sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$ the principal part (Hauptteil) of f in a,
- $\operatorname{res}(f, a) = c_1$ the residuum of f in a.

We say that f has a simple pole at a if $m = 1$.

Laurent separation

Lemma (9.7)

Let $f \in \mathcal{O}(U \setminus \{a\})$ $(a \in U)$. Then equivalent are:

- *f has a pole of order m in a,*
- *$\exists\, g \in \mathcal{O}(U)$ with $g(a) \neq 0$ and $f(z) = \frac{g(z)}{(z-a)^m}$ for all $z \in U \setminus \{a\}$,*
- *$\lim_{z \to 0}(z-a)^m f(z) \in \mathbb{C}^*$ exists.*

In this case: $\mathrm{res}\,(f,a) = \frac{g^{(m-1)}(a)}{(m-1)!}$*. If $m = 1$, then*

$$\mathrm{res}\,(f,a) = g(a) = \lim_{z \to a}(z-a)f(z).$$

We return to Example 8.11. For $a \in \mathbb{C}, 0 \leq s < t \leq \infty$, set

$$K_{s,t}(a) = \{z \in \mathbb{C};\ s < |z-a| < t\}.$$

Theorem (Laurent separation = 9.8)

Let $f \in \mathcal{O}(K_{s,t}(a))$. Then $\exists!\ f_0 \in \mathcal{O}(D_t(a)), f_\infty \in \mathcal{O}(\mathbb{C} \setminus \overline{D}_s(a))$ with

$$f = f_0 + f_\infty \text{ on } K_{s,t}(a), \qquad \lim_{|z| \to \infty} f_\infty(z) = 0.$$

Beweis von Satz 9.8

Beweis von Satz 9.8

Sei $f \in \mathcal{O}(K_{s,t}(a))$. Schreibe $D_t(a) = \bigcup_{s<r<t} D_r(a)$, $\mathbb{C} \setminus \bar{D}_s(a) = \bigcup_{s<r<t} \mathbb{C} \setminus \bar{D}_r(a)$.

$$\left.\begin{array}{l} f_0(z) \\ f_\infty(z) \end{array}\right\} := \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{f(\xi)}{\xi - z}\, d\xi \quad \left\{ \begin{array}{l} \text{für } z \in D_r(a) \\ \text{für } z \notin \bar{D}_r(a) \end{array}\right.$$

Lemma 8.4 $\Rightarrow$ f_0 (f_∞) holomorph auf $D_r(a)$ ($\mathbb{C} \setminus \bar{D}_r(a)$) und $\lim_{|z|\to\infty} f_\infty(z) = 0$

Allg. CIS $\Rightarrow$ $f_0(z)$, $f_\infty(z)$ unabhängig von der Wahl von r und (Beispiel 8.11)

$$f = f_0 - f_\infty \text{ auf } K_{s,t}(a)$$

Eindeutigkeit von f_0, f_∞:

Seien $f = f_0 + f_\infty = g_0 + g_\infty$ wie im Satz

$\Rightarrow \exists\, h \in \mathcal{O}(\mathbb{C})$ mit $h = \begin{cases} f_0 - g_0 & \text{auf } D_t(a) \\ g_\infty - f_\infty & \text{auf } \mathbb{C} \setminus \bar{D}_s(a) \end{cases}$

$\Rightarrow \lim_{|z|\to\infty} h(z) = 0$

$\Rightarrow$ (Liouville) $h = 0$.

Laurent expansion

Expanding f_0 (Nebenteil) and f_∞ (Hauptteil) into series

$$f_0(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \quad (|z-a| < t), \quad f_\infty(z) = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \quad (|z-a| > s)$$

one obtains:

Theorem (Laurent-Entwicklung = 9.9)

For $f \in \mathcal{O}(K_{s,t}(a))$ there are unique coefficients a_n $(n \in \mathbb{Z})$ with

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \text{ for all } z \in K_{s,t}(a).$$

The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi \quad (n \in \mathbb{Z}, s < r < t).$$

The series $\sum_{n=0}^{\infty} a_n(z-a)^n$ and $\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$ converge uniformly on compact subsets of $D_t(a)$ and $C \setminus \overline{D}_s(a)$, respectively.

Theorem 9.9: Proof

$\dot{D}_{1/s}(0) \to \mathbb{C},\ z \mapsto f_\infty(a + \frac{1}{z})$ extends to $F_\infty \in \mathcal{O}(D_{1/s}(0))$ with Taylor series

$$F_\infty(z) = \sum_{n=1}^{\infty} a_{-n} z^n \quad \text{(Riemansscher Hebbarkeitssatz)}$$

$$f_\infty(z) = F_\infty\left(\frac{1}{z-a}\right) = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \text{ for } |z-a| > s$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} =: \sum_{n\in\mathbb{Z}} a_n(z-a)^n$$

If $f(z) = \sum_{n\in\mathbb{Z}} b_n(z-a)^n$ for $z \in K_{s,t}(a)$, then the radius of convergence of

$$\sum_{n=0}^{\infty} b_n(z-a)^n \text{ is } \geq t \quad \left(\sum_{n=1}^{\infty} b_{-n} z^n \text{ is } \geq 1/s\right).$$

Hence the series $\sum_{n\in\mathbb{Z}} b_n(z-a)^n$ converges uniformly on cpct subsets of $K_{s,t}(a)$ and

$$\int_{\partial D_r(a)} \frac{f(\xi)}{(\xi-a)^{N+1}} d\xi = \sum_{n\in\mathbb{Z}} b_n \int_{\partial D_r(a)} (z-a)^{n-N-1} d\xi = 2\pi i\, b_N.$$

Laurent series

Definition

Let $f \in \mathcal{O}(K_{s,t}(a))$ with $0 \leq s < t \leq \infty$. The series expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z-a)^n$$

is called the Laurent series of f. We call

- $\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} = \sum_{n=-1}^{-\infty} a_n (z-a)^n$ the principal part of f
- $\sum_{n=0}^{\infty} a_n (z-a)^n$ the Nebenteil of f
- $\operatorname{res}(f,a) = a_{-1} = \frac{1}{2\pi i} \int\limits_{\partial D_r(a)} f(\xi) d\xi$ $(s < r < t)$ the residuum of f.

Examples

Let $f(z) = \frac{1}{z(z-1)^2}$. Then expanding f into its Laurent series yields

- $f(z) = \frac{1}{z} \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{z} \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=-1}^{\infty} (n+2) z^n$ on $K_{0,1}(0)$
- $f(z) = \frac{1}{(z-1)^2} \frac{1}{1+(z-1)} = \sum_{n=-2}^{\infty} (-1)^n (z-1)^n$ on $K_{0,1}(1)$.

How to recognize the singularity type from the Laurent series

Theorem

Let $f \in \mathcal{O}(U \setminus \{a\})$ ($a \in U \subset \mathbb{C}$ open) with Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z-a)^n \quad \text{near } a$$

Then

- *a is removable $\Leftrightarrow$ $a_n = 0 \; \forall n < 0$*
- *a is a pole of order m $\Leftrightarrow$ $a_{-m} \neq 0$ and $a_n = 0 \; \forall n < -m$*
- *a is essential $\Leftrightarrow$ $a_n \neq 0$ for infinitely many $n < 0$.*

Some ideas: If f extends to $g \in \mathcal{O}(U)$, then

$$a_n = \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{g(\xi)}{(\xi - a)^{n+1}} d\xi = 0 \; \forall n < 0.$$

If $f(z) - \sum_{k=1}^{m} c_{-k}(z-a)^{-k}$ extends to $g \in \mathcal{O}(U)$, then

$$f(z) = \sum_{k=-m}^{-1} c_k (z-a)^k + \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!}(z-a)^k$$

is the Laurent series of f at a. The third case follows by excluding the first two.

Residue theorem

Theorem

Let $f \in \mathcal{O}(U \setminus A)$ ($A \subset U$ discrete). If Γ is a cycle in $U \setminus A$ with $\mathrm{Int}(\Gamma) \subset U$, then

$$\frac{1}{2\pi i} \int_\Gamma f \, dz = \sum_{a \in A} \mathrm{ind}_\Gamma(a) \, \mathrm{res}(f, a) \quad \textit{(the sum is finite!)}.$$

Idea: The set $M = \{a \in A;\ \mathrm{ind}_\Gamma(a) \neq 0\} \subset \mathrm{Int}(\Gamma) \cup \mathrm{Sp}(\Gamma)$ is finite, say

$$M = \{a_1, \ldots, a_k\}.$$

Let $q_i(z) = \sum_{n=1}^{\infty} a_{in}(z - a_i)^{-n}$ be the principal part of f in a_i.

Then $f - (q_1 + \ldots + q_k)$ extends to $g \in \mathcal{O}(V)$ on $V = (U \setminus A) \cup \{a_1, \ldots, a_k\}$ and

$$\int_\Gamma f \, dz = \int_\Gamma \left(\sum_{i=1}^{k} q_i \right) dz = \sum_{i=1}^{k} \sum_{n=1}^{\infty} a_{in} \int_\Gamma \frac{1}{(z - a_i)^n} dz = \sum_{i=1}^{k} (2\pi i) \, \mathrm{ind}_\Gamma(a_i) \, \mathrm{res}(f, a_i).$$

First equality: CIS (8.10 (a)). Last equality: CIF for the derivatives (8.10 (c)).

Zeroes and poles

Definition

Let $f \in \mathcal{O}(U)$ and let a be a zero point or a pole for f. Then the order of f at a is

$$\mathrm{ord}_a(f) = \begin{cases} n, & \text{if } a \text{ is a zero of multiplicity } n \\ -n, & \text{if } a \text{ is a pole of order } n \end{cases}$$

We write $\mathrm{ord}_a(f) = 0$ if $a \in U$ is no zero of f and define $N_f = \overset{-1}{f}(\{0\})$.

Examples

Let $f \in \mathcal{O}(U)$. If a is a zero of finite multiplicity or a pole for f, then

$$\frac{f'}{f} \text{ has a simple pole at } a \text{ with } \mathrm{res}(\frac{f'}{f}, a) = \mathrm{ord}_a(f).$$

Indeed, if $f(z) = (z-a)^n g(z)$ $(n \in \mathbb{Z}^*)$ with $g(a) \neq 0$, then on $\dot{D}_r(a) \subset U$

$$\frac{f'(z)}{f(z)} = \frac{n(z-a)^{n-1}g(z)}{(z-a)^n g(z)} + \frac{(z-a)^n g'(z)}{(z-a)^n g(z)} \in \frac{n}{z-a} + \mathcal{O}(D_r(a)).$$

Argument principle

Simple facts: Let $M, N \subset \mathbb{C}$ and $f \in \mathcal{O}(U)$ be given.

- a accumulation pt. (Häufungspkt) of M
  $:\Leftrightarrow\ \forall r > 0 :\ \dot{D}_r(a) \cap M \neq \emptyset\ \Leftrightarrow\ \exists\, M \cap \{a\}^c \ni z_n \xrightarrow{(n\to\infty)} a$
- a accumulation pt. of $M \cup N$ $:\Leftrightarrow$ M accumulation pt. of M or N
- f not identically zero on any component of $U \overset{Id.Satz}{\Longleftrightarrow} N_f \subset U$ discrete

Theorem (Argument principle = 10.3)

Let $f \in \mathcal{O}(U \setminus A)$ ($A \subset U$ discrete) such that $N_f \subset U \setminus A$ is discrete and A consists of poles for f.

If Γ is a cycle in $U \setminus (A \cup N_f)$ with $\mathrm{Int}(\Gamma) \subset U$, then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz = \sum_{a \in N_f \cup A} \mathrm{ind}_\Gamma(a)\, \mathrm{ord}_a(f) \qquad \textit{(the sum is finite!)}.$$

If $A = \emptyset$, then the formula holds without A for each $f \in \mathcal{O}(U)$ with $N_f \subset U$ discrete and each cycle Γ in $U \setminus N_f$ with $\mathrm{Int}(\Gamma) \subset U$.

Arguement principle: Proof

Idea: Apply the residue theorem to $\frac{f'}{f} \in \mathcal{O}(U \setminus (A \cup N_f))$

Assume that there is an accumulation point a of $A \cup N_f$ in U.

Then a is an accumulation point of N_f in $U \setminus (U \setminus A) = A$.

$$\Rightarrow \quad \exists\, N_f \ni z_n \overset{(n \to \infty)}{\longrightarrow} a \in A$$

which is not possible, since a would be a pole for f.

Thus $\frac{f'}{f} \in \mathcal{O}(U \setminus (A \cup N_f))$ and $A \cup N_f \subset U$ is discrete.

$$\Rightarrow \text{ (Resiude theorem) } \frac{1}{2\pi i} \int_\Gamma \frac{f'}{f} dz = \sum_{a \in N_f \cup A} \mathrm{ind}_\Gamma(a) \mathrm{ord}_a(f)$$

for each cycle Γ in $U \setminus (A \cup N_f)$ with $\mathrm{Int}(\Gamma) \subset U$.

Counting zeroes and poles

Remark

If in addition $\mathrm{ind}_\Gamma(a) = 1\ \forall a \in \mathrm{Int}(\Gamma)$, then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = \sum_{a\in N_f\cap \mathrm{Int}(\Gamma)} \mathrm{ord}_a(f) + \sum_{a\in A\cap \mathrm{Int}(\Gamma)} \mathrm{ord}_a(f)$$

is the difference of the number of zeroes and poles in $\mathrm{Int}(\Gamma)$ counted with multplicity and order.

Theorem (Rouché = 10.8)

If $f, g \in \mathcal{O}(U)$ are not zero on any component of U, Γ is a cycle in U with

$$|f(z) - g(z)| < |f(z)|\ \forall\ z \in \mathrm{Sp}(\Gamma)$$

and $\mathrm{Int}(\Gamma) \subset U$, then

$$\sum_{a\in N_f} \mathrm{ind}_\Gamma(a)\mathrm{ord}_a(f) = \sum_{a\in N_g} \mathrm{ind}_\Gamma(a)\mathrm{ord}_a(g) \quad \textit{(both sums are finite!)}.$$

Rouché's theorem: Proof

By the argument principle with $A = \emptyset$

$$\sum_{a \in N_f} \operatorname{ind}_\Gamma(a)\operatorname{ord}_a(f) = \frac{1}{2\pi i}\int_\Gamma \frac{f'}{f}dz, \quad \sum_{a \in N_g} \operatorname{ind}_\Gamma(a)\operatorname{ord}_a(g) = \frac{1}{2\pi i}\int_\Gamma \frac{g'}{g}dz$$

With $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $f(\Gamma) = (f \circ \gamma_1, \ldots, f \circ \gamma_n)$

$$\int_\Gamma \frac{f'}{f}dz = \sum_{i=1}^n \int_{a_i}^{b_i} \frac{f'(\gamma_i(t))}{f(\gamma_i(t))}\gamma_i'(t)dt = \sum_{i=1}^n \int_{f\circ\gamma_i} \frac{1}{z}dz = (2\pi i)\ \operatorname{ind}_{f(\Gamma)}(0).$$

Hence it suffices to show that

$$\operatorname{ind}_{f(\Gamma)}(0) = \operatorname{ind}_{g(\Gamma)}(0).$$

This equality follows from the hypothesis that (Lemma 10.7)

$$|f(\gamma_i(t)) - g(\gamma_i(t))| < |f(\gamma_i(t))| \quad \forall i = 1, \ldots, n,\ t \in [a_i, b_i].$$

Examples

(a) If $\text{ind}_\Gamma(a) = 1$ for all $a \in \text{Int}(\Gamma)$ in Rouché's theorem then,

$$\sum_{a\in N_f\cap\text{Int}(\Gamma)} \text{ord}_a(f) = \sum_{a\in N_g\cap\text{Int}(\Gamma)} \text{ord}_a(g).$$

(b) For $g(z) = z^8 - z^3 + 5z - 2$ and $f(z) = 5z$,

$$|g(z) - f(z)| = |z^8 - z^3 - 2| \leq 4 < 5 = |f(z)| \text{ on } \partial D_1(0).$$

Hence g has exactly as many zeroes in $D_1(0)$ as f, this means one.

Theorem (10.12)

Let $f \in \mathcal{O}(G)$ ($G \subset \mathbb{C}$ domain). If $f - w_0$ has a zero of order $k \in \mathbb{N}^*$ at $z_0 \in G$, then $\forall r > 0\ \exists$ open neighbourhoods $U \subset D_r(z_0)$ of z_0, $W \subset \mathbb{C}$ of w_0 with

$$\#\Big(\{z \in U; f(z) = w\}\Big) = k \quad \forall w \in W \setminus \{w_0\} \quad \textit{and} \quad f(U) = W.$$

Identity thm. $\Rightarrow \exists r > 0 : w_0 \notin f(\overline{D}_r(z_0) \setminus \{z_0\})$ and $0 \notin f'(\overline{D}_r(z_0) \setminus \{z_0\})$

For $|w - w_0| < s$ small enough: $\quad |(f - w) - (f - w_0)| < |f - w_0| \quad \text{on} \quad \partial D_r(z_0)$

(Argument principle) $\Rightarrow \quad \#\Big(\{z \in D_r(z_0); f(z) = w\}\Big) = k \quad \forall w \in \dot{D}_s(w_0).$

Local invertibility

Theorem (10.14)

Let $f \in \mathcal{O}(U)$and $z_0 \in U$ with $f'(z_0) \neq 0$.

$\Rightarrow$ $\exists$ *open ngh.hoods $V \subset U$ of z_0, W of $f(z_0)$ such that $f: V \to W$ is biholomorphic.*

Proof. May suppose that U is connected (Replace U by the component of z_0). Set $w_0 = f(z_0)$.

$\Rightarrow$ $f - w_0$ has a zero of order 1 at z_0.

$\Rightarrow$ $\exists$ open ngh.hoods $V \subset U$ of z_0, W of $f(z_0)$ such that $f: V \to W$ is bijective.

$\Rightarrow$ (Satz v. d. Gebietstreue) $g = (f: V \to W)^{-1}$ is continuous

$$\frac{g(w+h) - g(w)}{h} = \frac{g(w+h) - g(w)}{f(g(w+h)) - f(g(w))} \overset{(h \to 0)}{\to} \frac{1}{f'(g(w))}$$

Corollary (10.15)

Let $f \in \mathcal{O}(U)$ be injective. Then $f(U) \subset \mathbb{C}$ is open and $f: U \to f(U)$ is biholomorphic.

Applications to improper real integrals

Aim: Use the residue theorem to calculate real integrals of the form

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx, \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\sin(ax)dx, \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\cos(ax)dx \quad (p, q \in \mathbb{C}[z], 0 \notin q(\mathbb{R}), a > 0)$$

Strategy: Calculate

$$\int_{\Gamma_1 \wedge \Gamma_R} \frac{p(z)}{q(z)} e^{iaz} dz$$

with $\Gamma_1(x) = x\ (-R \leq x \leq R)$ and $\Gamma_R(x) = Re^{ix}\ (0 \leq x \leq \pi)$. If

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{p(z)}{q(z)} e^{iaz} dz = 0,$$

then the residue theorem implies that with $f(z) = \frac{p(z)}{q(z)} e^{iaz}$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{iax} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{p(x)}{q(x)} e^{iax} dx = 2\pi i \sum_{\substack{w \in N_q \\ \mathrm{Im}\ w > 0}} \mathrm{res}\,(f, w)$$

provided the improper Riemann integral on the left-hand side exists.

$\deg(q) \geq \deg(p) + 2$

Recall: For $f, g : \mathbb{R} \to \mathbb{C}$ continuous,

- by definition $\int_{-\infty}^{\infty} f dx$ exists if $\lim_{R\to\infty} \int_0^R f dx$ and $\lim_{R\to\infty} \int_{-R}^0 f dx$ both exist or, equivalently, if $\forall \epsilon > 0 \exists r_\epsilon > 0$ such that $\forall r, s \in \mathbb{R}$ with $r_\epsilon < r < s$.

$$\left| \int_r^s f dx \right| < \epsilon \text{ und } \left| \int_{-s}^{-r} f dx \right| < \epsilon \quad \text{(Cauchy criterion)}.$$

- if $\int_{-\infty}^{\infty} |g(x)| dx$ exists and $|f(x)| \leq c|g(x)| \forall |x| > R$, then $\int_{-\infty}^{\infty} f dx$ exists.

Theorem (11.2)

For $p, q \in \mathbb{C}[z]$ with $\deg(q) \geq \deg(p) + 2$ and $0 \notin q(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{\substack{w \in N_q \\ \mathrm{Im}\ w > 0}} \mathrm{res}\left(\frac{p}{q}, w\right).$$

Idea: Show that $\exists R_0, c > 0$ such that $\|p/q\|_{\mathrm{Sp}(\Gamma_R)} \leq c/R^2 \forall R > R_0$. Then

$$\left| \int_{\Gamma_R} p/q dz \right| \leq L(\Gamma_R)\, \|p/q\|_{\mathrm{Sp}(\Gamma_R)} \leq (\pi R)\, c/R^2 \xrightarrow{(R\to\infty)} 0.$$

Example 11.3

Examples

We calculate

$$I = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx$$

The only poles of

$$f(z) = \frac{z^2}{1+z^4} = \frac{g(z)}{h(z)}$$

in the upper half plane are simple poles at $w_1 = e^{i\frac{\pi}{4}}$ and $w_2 = e^{i\frac{3}{4}\pi}$ with (Aufgabe 33(b))

$$\operatorname{res}(f, w_i) = \frac{g(w_i)}{h'(w_i)} = \frac{1}{4w_i}$$

By Theorem 11.2

$$I = 2\pi i(\operatorname{res}(f, w_1) + \operatorname{res}(f, w_2)) = \frac{\pi}{2} e^{i\frac{\pi}{2}}(e^{-i\frac{\pi}{4}} + e^{-i\frac{3}{4}\pi}) = \frac{\pi}{\sqrt{2}}.$$

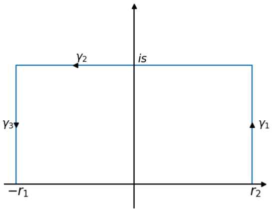
$\deg(q) \geq \deg(p) + 1$

Theorem (11.5)

If $p, q \in \mathbb{C}[z]$ with $\deg(q) \geq \deg(p) + 1$, $0 \notin q(\mathbb{R})$ and $a > 0$, then

$$\int_{-\infty}^{\infty} \underbrace{\frac{p(x)}{q(x)} e^{iax}}_{f(x)} dx = 2\pi i \sum_{\substack{w \in N_q \\ \mathrm{Im}\ w > 0}} \mathrm{res}\left(\frac{p(z)}{q(z)} e^{iaz}, w\right).$$

Ideas: Choose $r_1, r_2, s > 0$ with $\{w \in N_q; \mathrm{Im}\ w > 0\} \subset]-r_1, r_2[\times]0, s[$. Fix $\epsilon > 0$.



Using the residue theorem and choosing suitable $r_1, r_2 > 0$ and then $s > 0$ one obtains

$$\left| \int_{-r_1}^{r_2} f(x)dx - 2\pi i \sum_{\substack{w \in N_q \\ \mathrm{Im}\ w > 0}} \mathrm{res}\,(f, w) \right| = \left| - \int_{\gamma_1 \wedge \gamma_2 \wedge \gamma_3} f(z)dz \right| < \epsilon + \epsilon + \epsilon.$$

Theorem 11.5: Proof

- $|f(z)| = \left|\frac{p(z)}{q(z)} e^{iaz}\right| \underbrace{\leq}_{|z|\geq R_0} \frac{c}{|z|} e^{-a \operatorname{Im} z} \left(\leq \frac{c}{s} \text{ auf } \gamma_2\right)$
- $\gamma_1(t) = r_2 + ist \quad (0 \leq t \leq 1)$
- $\left|\int_{\gamma_1} f dz\right| = \left|\int_0^1 \frac{p}{q}(\gamma_1(t))\, e^{(iar_2 - ast)}\, isdt\right| \leq \frac{1}{a}\frac{c}{r_2} \underbrace{\int_0^1 e^{-ast}\, as\, dt}_{=1-e^{-as}\leq 1} \leq \frac{c}{ar_2}$
- $\left|\int_{\gamma_3} f dz\right| \leq \frac{c}{ar_1}$
- $\left|\int_{\gamma_2} f dz\right| \leq L(\gamma_2)\, \|f\|_{\mathrm{Sp}(\gamma_2)} \leq (r_1 + r_2)\, \frac{c}{s} < \epsilon$ for $s > s(r_1, r_2, \epsilon)$

Hence

$$\left|\int_{-r_1}^{r_2} f(x)dx - 2\pi i \sum_{\substack{w\in N_q \\ \operatorname{Im} w>0}} \operatorname{res}(f, w)\right| < 3\,\epsilon \quad \text{for } r_1, r_2 > r(\epsilon).$$

Example 11.6

Examples

We calculate

$$I = \int_{-\infty}^{\infty} \frac{x^3}{1+x^4} \sin(ax)dx \quad (a > 0).$$

The only poles of $f(z) = \frac{z^3 e^{iaz}}{1+z^4} = \frac{g(z)}{h(z)}$ in the upper half plane are simple poles at

$$w_1 = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}, w_2 = e^{i\frac{3}{4}\pi} = \frac{-1+i}{\sqrt{2}}$$

with residues (Aufgabe 33(b))

$$\text{res}(f, w_i) = \frac{g(w_i)}{h'(w_i)} = \frac{1}{4} e^{iaw_i}.$$

Theorem 11.5 yields

$$\int_{-\infty}^{\infty} f dx = \frac{2\pi i}{4} \left(e^{ia\frac{1+i}{\sqrt{2}}} + e^{ia\frac{-1+i}{\sqrt{2}}} \right) = \pi i\, e^{-\frac{a}{\sqrt{2}}} \cos\left(\frac{a}{\sqrt{2}}\right).$$

and hence $I = \text{Im}\left(\int_{-\infty}^{\infty} f dx\right) = \pi e^{-\frac{a}{\sqrt{2}}} \cos \frac{a}{\sqrt{2}}$.

Automorphism groups

Biholomorphic maps $f : U \to V$ ($U, V \subset \mathbb{C}$ open) are also called conformal mappings.

Definition

Let $U \subset \mathbb{C}$ be open. The automorphism group of U is (with composition as group operation)

$$\mathrm{Aut}(U) = \{f; f : U \to U \text{ biholomorphic}\}.$$

Examples

- $f : H_+ = \{z \in \mathbb{C};\ \mathrm{Im}\, z > 0\} \to D_1(0), f(z) = \frac{z-i}{z+i}$ is biholomorphic with inverse
$$g : D_1(0) \to H_+, g(z) = i\frac{1+z}{1-z}.$$
- The Moebius transformation (gebrochen lineare Transformation)
$$z \mapsto \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$$
defines a biholomorphic mapping
  (i) $f : \mathbb{C} \to \mathbb{C}$ if $c = 0$
  (ii) $f : \mathbb{C} \setminus \{-\frac{d}{c}\} \to \mathbb{C} \setminus \{\frac{a}{c}\}$ if $c \neq 0$

  with inverse given (up to the determinant) by the inverse matrix $z \mapsto \frac{dz-b}{-cz+a}$.

Automorphism group of $\mathbb{C}$

For $a \in \mathbb{C}^*, b \in \mathbb{C}$, define $f_{a,b} : \mathbb{C} \to \mathbb{C}, z \mapsto az + b$.

Theorem

$\mathrm{Aut}(\mathbb{C}) = \{f_{a,b}; a \in \mathbb{C}^* \text{ and } b \in \mathbb{C}\}$.

Idea: Let $f \in \mathrm{Aut}(\mathbb{C})$. Then 0 cannot be an essential singularity for

$$g : \mathbb{C}^* \to \mathbb{C}, g(z) = f\left(\frac{1}{z}\right).$$

since otherwise the open mapping principle (7.14) and Theorem 9.4 would imply that

$$f(\mathbb{C} \setminus \overline{D}_1(0)) \cap f(D_1(0)) = g(\dot{D}_1(0)) \cap f(D_1(0)) \neq \emptyset.$$

By Theorem 9.4 $\exists m \geq 1, c_1, \ldots, c_m \in \mathbb{C}$ such that 0 is a removable singularity for $f\left(\frac{1}{z}\right) - \sum_{k=1}^{m} \frac{c_k}{z^k}$.

$\Rightarrow \mathbb{C} \to \mathbb{C}, z \mapsto f(z) - \sum_{k=1}^{m} c_k z^k$ is bounded, hence constant by Liouville.

$\Rightarrow$ Since $f' \in \mathbb{C}[z]$ has no zero (chain rule), f is a polynomial with $\deg(f) = 1$.

Schwarz lemma

Obvious biholomorphic maps of the unit disc $\mathbb{D} = D_1(0)$ are:

- rotations $f(z) = e^{i\theta} z$ $(\theta \in \mathbb{R})$
- Moebius transformations $T_a(z) = \frac{z-a}{1-\bar{a}z}$ $\quad (a \in \mathbb{D})$ with inverse

$$T_{-a}(z) = \frac{z+a}{1+\bar{a}z}, \quad \begin{pmatrix} 1 & -a \\ -\bar{a} & 1 \end{pmatrix}^{-1} = \frac{1}{1-|a|^2} \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix}$$

Note that: $|T_a(z)| = \left| \frac{z-a}{(\overline{z-a})z} \right| = 1$ for $|z| = 1 \Rightarrow$ Max. principle $T_a(\mathbb{D}) \subset \mathbb{D}$.

Theorem (Schwarz lemma = 12.4)

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then

- $|f'(0)| \leq 1$
- $|f(z)| \leq |z| \; \forall \, z \in \mathbb{D}$.

If $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$, then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

Automorphism group of $\mathbb{D}$

Corollary (12.5)

$\mathrm{Aut}(\mathbb{D})$ *consists precisely of the mappings* $f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z} \quad (a \in \mathbb{D}, \theta \in \mathbb{R})$.

Idea: For $f \in \mathrm{Aut}(\mathbb{D})$, set $a = f^{-1}(0)$.

$\Rightarrow \; g = T_a \circ f^{-1} \in \mathrm{Aut}\,(\mathbb{D})$ with $g(0) = 0$

$\Rightarrow$ (Schwarz lemma for g and g^{-1}) $\;|g(z)| \leq |z| = |g^{-1}(g(z))| \leq |g(z)|$ for all z

$\Rightarrow \; \exists\theta \in \mathbb{R}$ with $f(z) = e^{-i\theta}g(f(z)) = e^{-i\theta}T_a(z)\; \forall z \in \mathbb{D}$.

Corollary (12.6)

Let $f \in \mathcal{O}(\mathbb{D})$ *be given with* $f(\mathbb{D}) \subset \mathbb{D}$*. Then:*

- $|f'(0)| \leq 1$ *with equality* $\Leftrightarrow \; \exists\, \theta \in \mathbb{R}: \; f(z) = e^{i\theta}z\; \forall z \in \mathbb{D}$.
- *If* $a, b \in \mathbb{D}$ *with* $f(a) = b$*, then* $|f'(a)| \leq \frac{1-|b|^2}{1-|a|^2}$ *with equality* $\Leftrightarrow \; \exists\, \theta \in \mathbb{R}: \; f(z) = T_{-b}(e^{i\theta}T_a(z))\; \forall z \in \mathbb{D}$.

Note: The automorphisms f of $\mathbb{D}$ with $f(0) = 0$ are precisely the rotations $f(z) = e^{i\theta}z$.

Riemannian sphere

Define $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (Riemannsche Zahlenkugel). The system

$$t = \{U; U \subset \mathbb{C} \text{ open}\} \cup \{\hat{\mathbb{C}} \setminus K; K \subset \mathbb{C} \text{ compact}\}$$

defines a topology on $\hat{\mathbb{C}}$ (or: turns X into a topological space), that is,

- $\varnothing, \hat{\mathbb{C}} \in t$,
- $U, V \in t \;\Rightarrow\; U \cap V \in t$,
- $(U_i)_{i \in I}$ arbitrary family in $t \;\Rightarrow\; \bigcup_{i \in I} U_i \in t$.

The sets in t are called the open subsets of $\hat{\mathbb{C}}$.

Definition (12.7)

A map $f : X \to Y$ between topopological spaces X and Y is called

- continuous at $x \in X \;\Leftrightarrow\; \forall$ open set $V \ni f(x)$ $\exists$ an open set $U \ni x$ such that $f(U) \subset V$
- continuous $\Leftrightarrow$ f is continuous at every $x \in X$ ($\Leftrightarrow \overset{-1}{f}(V) \subset X$ is open $\forall V \subset Y$ open)
- homeomorphism (Homömorphismus) $\Leftrightarrow$ f is bijective and f and f^{-1} are continuous.

A topological space X is called compact :$\Leftrightarrow$ Each open cover of X contains a finite subcover.

By definition $\lim_{n\to\infty} z_n = z$ in $\hat{\mathbb{C}}$:$\Leftrightarrow$ $\forall U \ni z$ open $\exists N \in \mathbb{N}$ with $z_n \in U\ \forall\ n \geq N$.

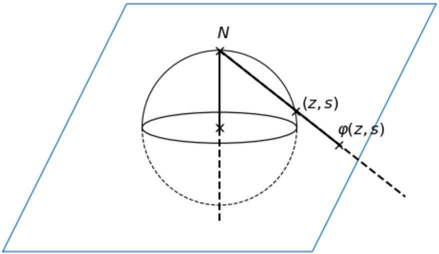
Beweis von Satz 9.8

$$S^2 = \{(z,s) \in \mathbb{C} \times \mathbb{R}; |z|^2 + s^2 = 1\} \subset \mathbb{R}^3 = \mathbb{C} \times \mathbb{R} \quad \text{(unit sphere)}$$

is a compact metric (in particular, topological) space. Let $N = (0,1) \in S^2$ be the north pole. One can define a homeomorphism $\varphi : S^2 \to \hat{\mathbb{C}}$ by setting $\varphi(N) = \infty$ and

$$\varphi(z,s) = N + t((z,s) - N) = (tz, 1 + t(s-1)) \text{ with } t = \frac{1}{1-s}$$

(intersection of the line through N and (z,s) with $\mathbb{C} = \mathbb{C} \times \{0\} \subset \mathbb{R}^3$).



Simple facts

Lemma

- $\hat{\mathbb{C}}$ *is a compact toplogical space (one-point compactificationof* $\mathbb{C}$*)*
- $\mathbb{C} \subset \hat{\mathbb{C}}$ *is open and dense*
- $U \subset \mathbb{C}$ *is open* $\Leftrightarrow\ \exists\, V \subset \hat{\mathbb{C}}$ *open with* $U = V \cap \mathbb{C}$
- $\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ z \mapsto \frac{1}{z}$ (*with* $\frac{1}{0} = \infty$ *and* $\frac{1}{\infty} = 0$) *is a homeomorphism.*
- *For* $z_n, z \in \mathbb{C}$: $\lim_{n\to\infty} z_n = z$ *in* $\mathbb{C}$ $\Leftrightarrow \lim_{n\to\infty} z_n = z$ *in* $\hat{\mathbb{C}}$
- *For* $z_n \in \mathbb{C}$: $\lim_{n\to\infty} z_n = \infty$ *in* $\hat{\mathbb{C}}$ $\Leftrightarrow \lim_{n\to\infty} |z_n| = \infty$

$\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ z \mapsto \frac{1}{z}$ is a homeomorphism, since $\mathrm{inv} : \mathbb{C}^* \to \mathbb{C}^* \hookrightarrow \hat{\mathbb{C}}$ is continuous and

$$\frac{1}{z}(D_\epsilon(0)) \subset \hat{\mathbb{C}} \setminus \overline{D}_{\frac{1}{\epsilon}}(0),\ \frac{1}{z}(\hat{\mathbb{C}} \setminus \overline{D}_{\frac{1}{\epsilon}}(0)) \subset D_\epsilon(0).$$

Function theory on $\hat{\mathbb{C}}$

Definition (12.10)

Let $U, V \subset \hat{\mathbb{C}}$ be open. A mapping $f : U \to \mathbb{C}$ is called holomorphic at $a \in U$ if

- $a \in \mathbb{C}$ and $f|_{U \cap \mathbb{C}}$ is holomorphic at a or
- $a = \infty$ and $\mathrm{inv}(U) \to \mathbb{C}, z \mapsto f(1/z)$ is holomorphic at 0.

A continuous map $f : U \to V$ is called holomorphic if $f \equiv \infty$ or

- $f^{-1}(\{\infty\}) \subset U$ is discrete and $f : U \setminus f^{-1}(\{\infty\}) \to \mathbb{C}$ is holomorphic

and biholomorphic if f is bijective and $f : U \to V, f^{-1} : V \to U$ are holomorphic.

For $f \in \mathcal{O}(U)$ ($U \subset \mathbb{C}$ open) we call ∞ a pole for f

$$:\Leftrightarrow \quad \exists s > 0 \text{ with } \{z \in \mathbb{C}; |z| > s\} \subset U \text{ and } \lim_{|z| \to \infty} |f(z)| = \infty.$$

Then $\infty \not\equiv f : U \to \hat{\mathbb{C}}$ ($U \subset \hat{\mathbb{C}}$ open) is holomorphic $\Leftrightarrow$ $A = f^{-1}(\{\infty\}) \subset U$ is discrete and

$$f : (U \setminus A) \cap \mathbb{C} \to \mathbb{C} \text{ is holomorphic with poles at each } a \in A.$$

Examples

- If $p, q \in \mathbb{C}[z]$ with $q \neq 0$ and $r(z) = \frac{p(z)}{q(z)}$ for $z \in \mathbb{C} \setminus N_q$, then

$$\hat{\mathbb{C}} \to \hat{\mathbb{C}}, z \mapsto \begin{cases} r(z) & ; z \in \mathbb{C} \setminus N_q \\ \lim\limits_{w \to z} r(w); & z \in \{\infty\} \cup N_q \end{cases}$$

  defines a holomorphic map.
- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$, the Moebius transformation $T_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, i.e., the unique continuous maps with

$$T_A(z) = \frac{az + b}{cz + d} \text{ for } z \in \mathbb{C} \text{ with } cz + d \neq 0$$

  is biholomorphic with $(T_A)^{-1} = T_{A^{-1}}$.

Theorem

$T : \mathrm{GL}(2, \mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}}), A \mapsto T_A$ is a surjective group homomorphism with

$$\mathrm{Ker}\ T = \mathbb{C}^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Riemann mapping theorem

Which domains in $\mathbb{C}$ are conformally equivalent to (= biholomorphic image of) $\mathbb{D}$?

Necessary condition: Let $f : \mathbb{D} \to G$ be biholomorphic and $g : G \to \mathbb{C}^*$ holomorphic.

$\Rightarrow$ ($\mathbb{D}$ simply connected) $\exists\, h \in \mathcal{O}(\mathbb{D})$ with $g \circ f = e^h \Rightarrow g = \exp(h \circ f^{-1})$

$\Rightarrow$ (Theorem 8.13) G is a simply connected domain.

Theorem (Riemann mapping theorem = 13.2)

Each simply connected domain $\mathbb{C} \neq G \subset \mathbb{C}$ is conformally equivalent to the unit disc $\mathbb{D}$.

Lemma

$G \subset \mathbb{C}$ domain, $\mathcal{O}(G) \ni f_n \xrightarrow{(n\to\infty)} f$ uniformly on compact subsets. Then:

If all f_n are injective, then f is injective or constant.

Idea: If $f \not\equiv const$, $a, b \in G, a \neq b \Rightarrow$ (Identity thm.)$\exists r > 0 : f(a) \notin f(\overline{D}_r(b) \setminus \{b\})$

$\Rightarrow \|(f_n - f_n(a)) - (f - f(a)\|_{\partial D_r(b)} < \min_{z \in \partial D_r(b)} |f(z) - f(a)|$ for $n \geq N$

$\Rightarrow$ (Rouché) $f - f(a)$ has no zero in $D_r(b)$.

Definition

Let $U \subset \mathbb{C}$ open. A subset $\mathcal{F} \subset \mathcal{O}(U)$ is called

- equicontinuous at $a \in U$ if $\forall \epsilon > 0 \ \exists \ \delta > 0$ with $f(D_\delta(a)) \ \subset D_\epsilon(f(a)) \ \forall f \in \mathcal{F}$
- bounded if $\sup\{\|f\|_K; \ f \in \mathcal{F}\} < \infty \ \forall K \subset U$ compact
- normal if each sequence in $\mathcal{F}$ has a locally uniformly convergent subsequence.

Lemma (13.5)

Each bounded set $\mathcal{F} \subset \mathcal{O}(U)$ is equicontinuous at each $a \in U$.

Idea: If $\overline{D}_r(a) \subset U$ and $M = \sup\{\|f\|_{\overline{D}_r(a)}; f \in \mathcal{F}\}$, then $\forall \ f \in \mathcal{F}$ and $|z - a| < r/2$

$$|f(z) - f(a)| \overset{\text{CIF}}{=} |\frac{1}{2\pi i} \int_{\partial D_r(a)} (\frac{f(\xi)}{\xi - z} - \frac{f(\xi)}{\xi - a}) d\xi|$$

$$= \frac{|z-a|}{2\pi} | \int_{\partial D_r(a)} \frac{f(\xi)}{(\xi - z)(\xi - a)} d\xi| \leq \frac{2}{r} M |z - a|$$

Lemma (13.6)

If $K \subset U$ is compact and $\mathcal{F} \subset \mathcal{O}(U)$ is equicontinuous at each $a \in K$.
$\Rightarrow \ \forall \epsilon > 0 \ \exists \delta > 0$ *such that* $\forall f \in \mathcal{F}: \ |f(z) - f(w)| < \epsilon \forall z, w \in K$ *with* $|z - w| < \delta$.

Idea: Otherwise for some $\epsilon > 0$ sequences $(z_n), (w_n)$ in K, (f_n) in $\mathcal{F}$ with

$$|z_n - w_n| < 1/n \quad \text{and} \quad |f(z_n) - f(w_n)| \geq \epsilon \ \ \forall n \in \mathbb{N}.$$

For suitable subsequences $\lim z_n = \lim w_n = z \in K$ exists and by equicontinuity at z

$$\epsilon \leq |f_n(z_n) - f_n(w_n)| \leq |f_n(z_n) - f_n(z)| + |f_n(z) - f_n(w_n)| \xrightarrow{n} 0$$

Lemma (13.7)

If $\{f_n; n \in \mathbb{N}\} \subset \mathcal{O}(U)$ is bounded and $\lim_{n\to\infty} f_n(z)$ exists $\forall z \in M \subset U$ dense

$\Rightarrow \ (f_n)_{n \in \mathbb{N}}$ *converges uniformly on all cpct. subsets of U to a function $f \in \mathcal{O}(U)$.*

Idea: Use 13.5 and 13.6 to show that the sequences $(f_n(z))_{n \in \mathbb{N}}$ satisfy the Cauchy condition uniformly for z in each compact subset $K \subset U$.

Beschränkte Familien sind normal

Theorem (Montel's theorem = 13.8)

Each bdd.sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{O}(U)$ has a locally uniformly convergent subsequence.

Idea: Let $M = \{a_k; k \in \mathbb{N}^*\} \subset U$ be dense. Choose subsequences

$$(f_{1,n})_{n\in\mathbb{N}} \text{ von } (f_n)_{n\in\mathbb{N}} \text{ st. } (f_n(a_1))_{n\in\mathbb{N}} \text{ converges}$$
$$(f_{2,n})_{n\in\mathbb{N}} \text{ von } (f_{1,n})_{n\in\mathbb{N}} \text{ st. } (f_{2,n}(a_2))_{n\in\mathbb{N}} \text{ converges}$$

and continue recursively. Then the diagonal sequence

$$(g_n)_{n\in\mathbb{N}} = (f_{n,n})_{n\in\mathbb{N}} \text{ converges pointwise on } M$$

and defines a locally uniformly convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ by Lemma 13.7.

Alternatively one can prove Montel's theorem using the equicontinuity of bounded subsets $\mathcal{F} \subset \mathcal{O}(U)$ and Arzela-Ascoli's theorem from topology.

Riemann mapping theorem: Proof

Let $\mathbb{C} \neq G \subset \mathbb{C}$ be a simply connected domain. Show successively:

(1) $\Sigma := \{f \in \mathcal{O}(G);\ f \text{ is injective with } f(G) \subset \mathbb{D}\} \neq \emptyset$.

(2) For each $g \in \Sigma$ with $g(G) \neq \mathbb{D}$ and each $a \in G\ \exists\ f \in \Sigma$ with $|f'(a)| > |g'(a)|$.

(3) Fix $a \in G$. Choose a sequence $(g_n)_{n\in\mathbb{N}}$ in Σ with

$$\lim_{n\to\infty} |g_n'(a)| = s := \sup\{|g'(a)|; g \in \Sigma\}.$$

May suppose (Montel's theorem) $(g_n)_{n\in\mathbb{N}} \stackrel{n\to\infty}{\longrightarrow} g \in \mathcal{O}(G)$ uniformly on cpct. subsets

$\Rightarrow$ (Weierstraß' theorem) $|g'(a)| = \lim_{n\to\infty} |g_n'(a)| = s \in (0, \infty)$

$\Rightarrow$ g is injective with $g(G) \subset \overline{\mathbb{D}}$

$\Rightarrow$ (Maximum principle) $g \in \Sigma$ and by Step 2 g is surjective.

$\Rightarrow$ (Holomorhic invertibility) $g : G \to \mathbb{D}$ is biholomorphic.

Wichtiges für die Klausur

- CRDG'en, harmonische Fkt'en, Bedingungen für Konstanz (Satz 2.19.)
- Potenzreihen, Konvergenzradius, Taylorentwicklung
- Existenz von Stammfunktionen
- Cauchyscher Integralsatz und Formel für Kreise, konvexe Gebiete
- Grundprinzipien: Liouville, Nullstellen (Vielfachheit), Identitätssatz, Maximumprinzip, Riemannscher Hebbarkeitssatz, Gebietstreue
- Allgemeiner CIS und CIF, einfacher Zusammenhang, holomorphe Logarithmen
- Isolierte Singularitäten: Charakterisierungen, Ordnung von Polstellen, Berechnung von Residuen (Lemma 9.7 + Aufg.33(b)) und Laurentreihen
- Residuensatz, Berechnung von Integralen (komplexe und reelle), Rouché
- Konforme Abbildungen von $\mathbb{C}$, $\mathbb{D}$, Schwarzsches Lemma, Moebiustransform.en

Schutzmasken mitbringen! Corona-Regeln für die Klausur beachten (Homepage)!

Viel Glück!!!