SAARLAND UNIVERSITY
FACULTY – MATH
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Exercises for the lecture Complex Analysis

Summer term 2020

Sheet 1 Deadline: Thursday, 05.14.2020, before the lecture

Exercise 1 (4 Points)

Let $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}^*$. Show that the equation

$$z^n = a$$

has exactly n distinct solutions in $\mathbb{C}$. Determine this solutions in polar coordinates.

Exercise 2 (1+1+2+2=6 Points)

For non-empty subsets M, N of $\mathbb{C}$ we call

$$d(M, N) = \inf\{|a - b|; a \in M, b \in N\}$$

the distance of M and N. Show:

(a) If M is compact, N closed and $M \cap N = \varnothing$, then $d(M, N) > 0$.

(b) There are non-empty, closed sets $M, N \subseteq \mathbb{C}$ with $M \cap N = \varnothing$ and $d(M, N) = 0$.

(c) If $a \in \mathbb{C}$, $r > 0$ and $K \subseteq D_r(a)$ is compact, then there is a real number $s \in \mathbb{R}$ with $0 < s < r$ and $K \subseteq D_s(a)$.

(d) If $U \subsetneq \mathbb{C}$ is open and $\varnothing \neq K \subseteq U$ compact, then for $0 < \epsilon < d(K, U^c)$

$$K_\epsilon = \{z \in \mathbb{C}; \exists w \in K \text{ with } |z - w| \leq \epsilon\}$$

is a compact subset of U.

Exercise 3 (2+2=4 Points)

Let $G_1, G_2 \subseteq \mathbb{C}$ be non-empty domains. Show:

(a) The set

$$G_1 - G_2 = \{z - w; z \in G_1, w \in G_2\}$$

is a domain in $\mathbb{C}$.

(b) The set $G_1 \cup G_2$ is a domain in $\mathbb{C}$ if and only if $G_1 \cap G_2 \neq \varnothing$.

(please turn over)

Exercise 4 **($3 + 1 + 3 \times 1 = 7$ Points)**

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be $\mathbb{R}$-linear and let $C \in M(2, \mathbb{R})$ be the representing matrix for T with respect to the canonical basis of $\mathbb{R}^2$. Show that the following statements are equivalent:

  (i) T is $\mathbb{C}$-linear,

  (ii) there exists a complex number $\lambda \in \mathbb{C}$ with $T(z) = \lambda z$ for all $z \in \mathbb{C}$,

  (iii) there exist real numbers $a, b \in \mathbb{R}$ with $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be totally differentiable at a point $(x_0, y_0) \in \mathbb{R}^2 = \mathbb{C}$. Show that the following statements are equivalent:

  (i) $Df(x_0, y_0) : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{C}$-linear,

  (ii) $\frac{\partial \mathrm{Re} f}{\partial x}(x_0, y_0) = \frac{\partial \mathrm{Im} f}{\partial y}(x_0, y_0)$ and $\frac{\partial \mathrm{Im} f}{\partial x}(x_0, y_0) = -\frac{\partial \mathrm{Re} f}{\partial y}(x_0, y_0)$.

Show that the following functions are totally differentiable as maps $\mathbb{R}^2$ from $\mathbb{R}^2$, and determine all points $z \in \mathbb{C}$ at which their total differentials are $\mathbb{C}$-linear.

(c) $g : \mathbb{C} \to \mathbb{C}, g(z) = z^2$,

(d) $h : \mathbb{C} \to \mathbb{C}, h(z) = \overline{z}$,

(e) $k : \mathbb{C} \to \mathbb{C}, k(z) = z|\overline{z}|^2$.

You may submit the solutions for the exercise sheets in groups up to three participants, belonging to the same tutorial group. But please don't meet in one person. To be admitted to the exam, you need 50 % of the points achievable in the homework assignments. Please send your solutions via E-mail in the form of a single pdf-file to your tutor. ***Two of the exercises will be corrected: this time Exercise 4 plus one exercise chosen randomly (the same for all students).***

You can also find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre