§ 10: Stone-Weierstrass Theorem

For a top. space (X,τ), let

$$C_{\mathbb{R}}(X) = \{f: X \to \mathbb{R};\ f \text{ continuous}\} \text{ and } C(X) := C(X,\mathbb{C}) = \{f: X \to \mathbb{C};\ f \text{ continuous}\}.$$

be equipped with pointwise addition/multiplication, $C_{\mathbb{R}}(X)$ is an $\mathbb{R}$-algebra, $C(X)$ a $\mathbb{C}$-algebra.
If X is compact, $\|\cdot\|_X$ endows them with a norm topology (which is even a Banach algebra norm)

10.1 Lemma

Let (X,τ) be a compact (Hausdorff) top. space and $V \subseteq C_{\mathbb{R}}(X)$ a linear subspace such that

(i) V contains the constants

(ii) V separates the points, i.e. $x,y \in X, x \neq y \Rightarrow \exists f \in V: f(x) \neq f(y)$ ⌜$\Rightarrow (X,\tau)$ is Hausdorff⌟

(iii) $f,g \in V \Rightarrow \underbrace{\min(f,g)}_{\frac{f+g-|f-g|}{2}}, \underbrace{\max(f,g)}_{\frac{f+g+|f-g|}{2}} \in V$

Then, $\overline{V} = C_{\mathbb{R}}(X)$ (w.r.t. $\|\cdot\|_X$).

10.2 Lemma:

For every $R>0$, there is a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{R}[x]$ with $q_n(0)=0$ for all $n \in \mathbb{N}$ and

$$\sup_{t\in[0,R]} |\sqrt{t} - q_n(t)| \xrightarrow{n\to\infty} 0.$$

10.3 Theorem (real version of Stone-Weierstrass Theorem)

Let (X,τ) be a compact (Hausdorff) top. space and $A \subseteq C_{\mathbb{R}}(X)$ a subalgebra such that:

$$\forall x,y \in X,\ x \neq y: \forall \alpha,\beta \in \mathbb{R}\ \exists f \in A: f(x) = \alpha,\ f(y) = \beta.$$

Then, $\overline{A} = C_{\mathbb{R}}(X)$.
This holds in particular if A fulfills i) and ii) from 10.1.

10.4 Theorem (complex version of Stone-Weierstrass Theorem)

Let (X,τ) be a compact (Hausdorff) top. space and $A \subseteq C(X)$ a subalgebra such that

(i) A contains the constants

(ii) A separates the points of X

(iii) $f \in A \Rightarrow \overline{f} \in A$

Then, $\overline{A} = C(X)$.

10.5 Corollary

If $K \subseteq \mathbb{R}^n$ is compact, in particular, $\mathbb{R}[x_1,\dots,x_n]|_K \subseteq C_{\mathbb{R}}(K)$ and $\mathbb{C}[x_1,\dots,x_n]|_K \subseteq C(K)$ are dense.

10.6 Definition

Let (X,τ) be a locally compact Hausdorff space. A function $f \in C_{\mathbb{R}}(X) \cup C(X)$ is said to vanish at infinity if

$$\forall \varepsilon > 0: \exists K \subseteq X \text{ compact}: |f(x)| < \varepsilon \text{ for all } x \in X\setminus K.$$

We write

$$C_0^{\mathbb{R}}(X) = \{f \in C_{\mathbb{R}}(X);\ f \text{ vanishes at infinity}\} \text{ and } C_0(X) = \{f \in C(X);\ f \text{ vanishes at infinity}\}.$$

⌜Note: For a function $f: V \to \mathbb{K}$ on a normed space V, one may demand $\lim_{\|x\|\to\infty} f(x) = 0$. These notions differ!⌟

10.7 Lemma

Let (X,τ) be a locally compact Hausdorff space, $\hat{X}$ its one-point-compactification. Then, we have

$$C_0(X) = \{f|_X;\ f \in C(\hat{X}) \text{ with } f(\infty) = 0\} \text{ and } C_0^{\mathbb{R}}(X) = \{f|_X;\ f \in C_{\mathbb{R}}(\hat{X}) \text{ with } f(\infty) = 0\}.$$

Proof:

We only prove the complex version.
First, let $\tilde{f} \in C_0(X)$. Then, $f: \hat{X} \to \mathbb{C}$, $f(x) = \begin{cases} \tilde{f}(x), & x \in X \\ 0, & x = \infty \end{cases}$ is an extension of $\tilde{f}$.
f continuous: In $x \in X$, $X \subseteq \hat{X}$ is open with $x \in X$, $f|_X = \tilde{f}$ continuous. In ∞: Let $\varepsilon > 0$. By def, there is $K \subseteq X$ compact: $\tilde{f}(X\setminus K) \subseteq D_\varepsilon(0)$. Since $\hat{X}\setminus K \in \mathcal{U}(\infty)$ with $f(\hat{X}\setminus K) \subseteq D_\varepsilon(0)$, f is continuous in ∞.
For $f \in C(\hat{X})$ with $f(\infty) = 0$, $\tilde{f} = f|_X$ is continuous. For $\varepsilon > 0$, there is $U \in \mathcal{U}(\infty)$ with $f(U) \subseteq D_\varepsilon(0)$.
$\Rightarrow \exists K \subseteq X$ compact: $\tilde{f}(X\setminus K) \subseteq f(\hat{X}\setminus K) \subseteq f(U) \subseteq D_\varepsilon(0)$. q.e.d.

10.8 Theorem (Stone-Weierstraß for $C_0^{\mathbb{R}}(X)$ / $C_0(X)$)

Let (X,τ) be a locally compact Hausdorff space and $A \subseteq C_0^{\mathbb{R}}(X)$ ⌜$C_0(X)$⌟ a subalgebra such that

(i) A separates the points of X,

(ii) for every $x \in X$, there is $f \in A$ with $f(x) \neq 0$,

⌜(iii) $f \in A \Rightarrow \overline{f} \in A$⌟

Then, $A \subseteq C_0^{\mathbb{R}}(X)$ ⌜$C_0(X)$⌟ is dense w.r.t. the supremum norm.

Proof:

For $f \in C_0^{\mathbb{R}}(X)$, let $\hat{f} \in C_{\mathbb{R}}(\hat{X})$ be defined as in the proof of Lemma 10.7.

$\Rightarrow A_1 = \{\hat{f} + \alpha 1;\ f \in A,\ \alpha \in \mathbb{R}\} \subseteq C_{\mathbb{R}}(\hat{X})$

is a subalgebra which contains the constants and separates the points of $\hat{X}$.

$\overset{10.3}{\Rightarrow} A_1 \subseteq C_{\mathbb{R}}(\hat{X})$ dense $\Rightarrow \forall f \in C_0^{\mathbb{R}}(X)$: $\exists$ sequences $(f_n)_{n\in\mathbb{N}}$ in A, $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$:

$$\hat{f}_n + \alpha_n 1 \xrightarrow[\|\cdot\|_{\hat{X}}]{n\to\infty} \hat{f}$$

Since $\alpha_n = (\hat{f}_n + \alpha_n 1)(\infty) \xrightarrow{n\to\infty} \hat{f}(\infty) = 0$, we conclude

$$\hat{f}_n + \alpha_n 1 - \alpha_n 1 \xrightarrow[\|\cdot\|_{\hat{X}}]{n\to\infty} \hat{f}$$

$$\Rightarrow f_n = \hat{f}_n|_X \xrightarrow[\|\cdot\|_X]{n\to\infty} \hat{f}|_X = f.$$

Complex version: analogously from 10.4 q.e.d.