§ 13 : Topological groups

13.1 Definition

Let $(G,\circ)$ be a group and τ a topology on G. The triple $(G,\circ,\tau)$ [or, shorter G] is called a topological group if the mappings

$\circ : G\times G \to G, (g,h)\mapsto g\circ h$ (where $G\times G$ is equipped with the product topology)

and

$i: G\to G,\ g\mapsto g^{-1}$

are continuous

13.2 Remark

a) It is easy to show that a group $(G,\circ)$ with a topology τ is a topological group if and only if the mapping $G\times G\to G, (g,h)\mapsto gh^{-1}$ is continuous.

b) For a topological group G and $g\in G$, the mappings $k_g: G\to G, h\mapsto ghg^{-1}$, $i: G\to G, g\mapsto g^{-1}$, $\ell_g: G\to G, h\mapsto gh$, $r_g: G\to G, h\mapsto hg$, are homeomorphisms.

13.3 Examples

a) Every group equipped with the discrete (or indiscrete) topology is a topological group.

b) $(\mathbb{R}^n,+)$, $(\mathbb{C}^n,+)$, $(\mathbb{R}\setminus\{0\},\cdot)$, $(\mathbb{C}\setminus\{0\},\cdot)$, $(S_1,\cdot)$ with the topology induced by $|\cdot|$ are topological groups. $\quad S_1 = \{x\in\mathbb{C};\ |x|=1\}$

c) For $n\in\mathbb{N}$, $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, the non-abelian groups
$GL_n(\mathbb{K}) = \{A\in\mathbb{K}^{n\times n};\ A \text{ invertible}\}$, $O_n(\mathbb{K}) = \{A\in GL_n(\mathbb{K}),\ A^{-1}=\bar{A}^T\}$, ...
with matrix multiplication as group operations are top. groups if equipped with their norm topology.

d) $(\mathbb{Q},+)$ with its norm topology is a topological group, but not a topological manifold.

(1) { e) Is there a group operation $\circ$ on $([0,1],\tau_{|\cdot|})$ such that $([0,1],\circ,\tau_{|\cdot|})$ is a topological group?

13.4 Lemma / Definition

Let $(G,\circ,\tau)$ be a topological group. Then, for every $g_1,g_2\in G$, there is a homeomorphism $h: G\to G$ with $h(g_1)=g_2$. We call a top. space with this property homogeneous.

Proof

For $g_1,g_2\in G$, the function $G\to G$, $h\mapsto hg_1^{-1}g_2$ is a homeomorphism which maps g_1 to g_2. q.e.d.

13.5 Remark

a) There is no homeomorphism $h: ([0,1],\tau_{|\cdot|})\to([0,1],\tau_{|\cdot|})$ with $h(1)=\tfrac12$ since $([0,1[,\tau_{|\cdot|})$ is connected while $h([0,1[)$ would be $[0,\tfrac12[\,\cup\,]\tfrac12,1]$.

(2) { **13.6 Lemma**

Let $(G,\circ,\tau)$ be a topological group, $H\subseteq G$ a subgroup.

a) $(H,\circ|_{H\times H},\tau|_H)$ is a topological group.

b) The set of left cosets (also right cosets)

$G/H = \{gH,\ g\in G\}$ $\quad (H\backslash G = \{Hg,\ g\in G\})$

equipped with the final topology induced by $q: G\to G/H, g\mapsto gH$ is homogeneous.

Proof:

a) follows since restrictions of cont. maps are continuous.

b) For $g_1H, g_2H\in G/H$, the function $\Phi: G/H\to G/H$, $gH\mapsto g_2g_1^{-1}gH$ is continuous since $\Phi\circ q = q\circ \ell_{g_2g_1^{-1}}$ is continuous and a homeomorphism since $\Phi^{-1}\circ q = q\circ \ell_{g_1g_2^{-1}}$.

q.e.d.

13.7 Definition

A function $f: G\to H$ between topological groups is called topological homomorphism (epimorphism, monomorphism, isomorphism) if it is a continuous group homomorphism (which is also surjective, injective, resp. bijective with continuous inverse function).

13.8 Remark

a) $(\mathbb{R},+)$ and $(\mathbb{R}^2,+)$ are isomorphic as groups (consider $\mathbb{Q}$-vector space bases for both), but not as topological groups ($\mathbb{R}^2\setminus\{0\}$ is connected, $\mathbb{R}\setminus\{p\}$ not connected for all $p\in\mathbb{R}$)

b) $(\mathbb{Z},\mathcal{P}(\mathbb{Z}))$ and $(\mathbb{Z}\times\mathbb{Z}/2,\mathcal{P}(\mathbb{Z}\times\mathbb{Z}/2))$ are isom. as top. spaces (i.e. homeomorphic), but not as (top.) groups (consider the element $(0,1)$ of order 2 in the latter).

13.9 Theorem

Let G,H be topological groups, $\mathcal{U}$ a neighbourhood base at the neutral element e_G of G. Then, we have

a) For $g\in G$, the sets $g\mathcal{U}=\{gU,\ U\in\mathcal{U}\}$ and $\mathcal{U}g=\{Ug;\ U\in\mathcal{U}\}$ are neighbourhood bases at g.
In particular, a group homom. is a topological group homomorphism if and only if it is continuous at e_G.

b) For $O\subseteq G$ open, $M\subseteq G$, $g\in G$, the sets $\underbrace{gO}_{\{gh,\ h\in O\}}, Og, \underbrace{O^{-1}}_{\{h^{-1},\ h\in O\}}, \underbrace{MO}_{\{h_1h_2,\ h_1\in M,\ h_2\in O\}}, OM\subseteq G$ are open.

c) For $A \subseteq G$ closed, $M \subseteq G$, $g \in G$, the sets gA, Ag, MA, AM, $A^{-1} \subseteq G$ are closed.
d) For $A, B \subseteq G$ compact, the sets $AB, A^{-1} \subseteq G$ are compact.
e) For $A \subseteq G$, we have

$$\overline{A} = \bigcap_{U \in \mathcal{U}(e_G)} AU = \bigcap_{U \in \mathfrak{U}} AU = \bigcap_{U \in \mathcal{U}(e_G)} UA = \bigcap_{U \in \mathfrak{U}} UA.$$

Proof:
a) For $U \in \mathcal{U}(g)$, we have $g^{-1}U \in \mathcal{U}(e_G)$ by 13.2b)
$\Rightarrow \exists V \in \mathfrak{U}: e_G \in V \subseteq g^{-1}U$
$\overset{13.2b)}{\Rightarrow} g \in gV \subseteq U$ with $gV \in g\mathfrak{U}$. Analogously for $\mathfrak{U}g$.
For $g \in G$, $\varphi: G \to H$ group hom, $V \in \mathcal{U}(\varphi(g))$, there is $V_0 \in \mathcal{U}(e_H)$ with $\varphi(g)V_0 \subseteq V$. Let $U_0 \in \mathcal{U}(e_G)$ with $\varphi(U_0) \subseteq V_0$. We conclude that $U = gU_0 \in \mathcal{U}(g)$ with $\varphi(U) \subseteq \varphi(g)\varphi(U_0) \subseteq \varphi(g)V_0 \subseteq V$.
b), c) follow from 13.2b) and $ME = \bigcup_{m \in M} mE$, $EM = \bigcup_{m \in M} Em$ for all $M, E \subseteq G$.
d) follows by def., since $A \times B \subseteq G \times G$ is comp. by Tychonoff (and 4.10c))
e) We only prove $\overline{A} = \bigcap_{U \in \mathfrak{U}} AU$, the rest is analogous/obvious.
"$\subseteq$" Let $g \in \overline{A}$, $U \in \mathfrak{U}$.
$\overset{a)}{\Rightarrow} gU^{-1} \in \mathcal{U}(g) \Rightarrow \exists h \in U^{-1}: gh \in A \Rightarrow g = ghh^{-1} \in AU$.
"$\supseteq$": Let $g \in \bigcap_{U \in \mathfrak{U}} AU$, $V \in \mathcal{U}(g) \Rightarrow V^{-1}g \in \mathcal{U}(e_G) \Rightarrow \exists U_0 \in \mathfrak{U}: U_0 \subseteq V^{-1}g$
$\overset{g \in AU_0}{\Rightarrow} \exists a \in A, v \in V: g = av^{-1}g \Rightarrow a = v \in A \cap V \Rightarrow g \in \overline{A}$.
q.e.d.

(3) }
13.11 Theorem:
Let G be a topological group, $F \subseteq G$ closed, $g \in G \setminus F$. Then, there are disjoint open sets $U, V \subseteq G$ such that $g \in U$, $F \subseteq V$. If G is T1 in addition (i.e. $\forall x, y \in G$, $x \neq y$: $\exists U \in \mathcal{U}(x)$, $V \in \mathcal{U}(y)$: $x \notin V$, $y \notin U$), G is regular (it is, in fact even completely regular in this case)!

Proof.
Let $F \subseteq G$ be closed, $g \in G \setminus F =: U_0$
$\Rightarrow g^{-1}U_0 \in \mathcal{U}(e_G)$. Choose $V_0 \in \mathcal{U}(e_G)$ with $V_0^{-1}V_0 \subseteq g^{-1}U_0$.
$\overset{13.9e)}{\Rightarrow} \overline{V_0} \subseteq V_0^{-1}V_0 \subseteq g^{-1}U_0$.
$\Rightarrow g\overline{V_0} \subseteq G$ closed with $g \in g\overline{V_0}$, $g\overline{V_0} \subseteq U_0$
$\Rightarrow U = G \setminus g\overline{V_0} \subseteq G$, $V = g\,\mathrm{Int}(V_0) \subseteq G$ open, disjoint with $F \subseteq U$, $g \in V$.
Suppose, G is T1 in addition, let $g, h \in G$ with $g \neq h$.
$\Rightarrow gh^{-1} \neq e_G \overset{G\ T1}{\Rightarrow} G \setminus \{gh^{-1}\} \in \mathcal{U}(e_G) \Rightarrow \exists V \in \mathcal{U}(e_G): V^{-1}V \subseteq G \setminus \{gh^{-1}\}$.
1A: $\exists z \in Vg \cap Vh \Rightarrow z = vg = wh$ for some $v, w \in V$
$\Rightarrow gh^{-1} = v^{-1}w \in V^{-1}V \subseteq G \setminus \{gh^{-1}\}$ $\lightning$
$\Rightarrow Vg \in \mathcal{U}(g)$, $Vh \in \mathcal{U}(h)$ are disjoint. q.e.d.

13.12 Lemma
For a top. group G and $A, B \subseteq G$, $g, h \in G$, we have
$\overline{A}\,\overline{B} \subseteq \overline{AB}$, $(\overline{A}^{-1}) = \overline{A^{-1}}$, $g\overline{A}h = \overline{gAh}$.
If $F \subseteq G$ is a subgroup, so is $\overline{F}$. If $N \subseteq G$ is a normal subgroup (i.e. $gNg^{-1} = N\ \forall g \in G$), so is $\overline{N}$. Furthermore, every open subgroup $F \subseteq G$ is also closed.
Proof.
The assertions about closures follow from Exercises 18 & 28 and remark 11.2.
The rest follows from the first part (note that $F \subseteq G$ is a subgroup if and only if $FF^{-1} \subseteq F$)
Finally, if $F \subseteq G$ is an open subgroup, so is

$$G \setminus F = \bigcup_{g \in G \setminus F} gF. \qquad \text{q.e.d.}$$

13.13 Theorem:
Let G be a topological group, $H \subseteq G$ a subgroup. As before (cf 13.6), let G/H be equipped with the quotient topology. Then, we have
a) G/H is discrete if and only if $H \subseteq G$ is open.
b) G/H is T1 if and only if $H \subseteq G$ is closed.
c) If $H \subseteq G$ is a normal subgroup, G/H is a top. group. It is regular if and only if $H \subseteq G$ is closed.
Proof
a) "$\Rightarrow$": $q^{-1}(\{e_G H\}) = H \subseteq G$ open "$\Leftarrow$": $H \subseteq G$ open $\Rightarrow \forall g \in G: gH \subseteq G$ open $\Rightarrow \forall g \in G: \{gH\} = q(gH) \subseteq G/H$ is open.
b) G/H T_1 $\Leftrightarrow \forall g \in G: \{gH\} \subseteq G/H$ closed $\Leftrightarrow \forall g \in G: \underbrace{gH}_{q^{-1}(\{gH\})} \subseteq G$ closed $\overset{11.2b)}{\Leftrightarrow} H \subseteq G$ closed
"$\Rightarrow$" as above
"$\Leftarrow$": choose compl. of one-point sets cf. ex. 35b)
c) We show that $G/H \times G/H \to G/H$, $(g_1H, g_2H) \mapsto g_1g_2^{-1}H$ is continuous (cf. 11.2a))
Let $g, h \in G$, $\tilde{W} \in \mathcal{U}(gh^{-1}H) \Rightarrow W = q^{-1}(\tilde{W}) \in \mathcal{U}(gh^{-1}) \Rightarrow \exists U \in \mathcal{U}(g), V \in \mathcal{U}(h): UV^{-1} \subseteq W$

Let $\tilde{U} = \{uH, u \in U\}, \tilde{V} = \{vH, v \in V\}, u \in U, v \in V$

$\Rightarrow \underset{uH \cdot v^{-1}H}{\underbrace{uv^{-1}H}_{=}} \in q(UV^{-1}) \subseteq q(W) \underset{q \text{ surjective}}{\overset{\uparrow}{=}} W$, thus $\tilde{U}\tilde{V}^{-1} \subseteq \tilde{W}$.

That G/H is regular if and only if $H \subseteq G$ is closed then follows from b) and 13.11.

q.e.d.

13.14 Theorem

Let $\varphi: G \to H$ be an open top. epimorphism between top. groups.

Then, $\varphi': G/\ker\varphi \to H, g\ker\varphi \mapsto \varphi(g)$ is a top isomorphism.

Proof:

That φ' is an algebraic isomorphism is a result from group theory.

Denote by q the quotient map $q: G \to G/\ker\varphi$.

Let $V \subseteq H$ open $\Rightarrow \varphi^{-1}(V) \subseteq G$ open $\Rightarrow q(\varphi^{-1}(V)) \subseteq G/\ker\varphi$ open.

$q(\varphi^{-1}(V)) = (\varphi'^{-1})(V)$

Now, let $U \subseteq G/\ker\varphi$ be open

$\Rightarrow q^{-1}(U) \subseteq G$ open $\Rightarrow \varphi'(U) = \varphi(q^{-1}(U)) \subseteq H$ open q.e.d.

13.15 Remark

We cannot forgo the cond.

(1) c) For a top. space X, the group

$$\mathrm{Hom}(X) = \{ f: X \to X \, ; \, f \text{ is a homeomorphism} \} \subseteq X^X$$

equipped the product topology ($\hat{=}$ top. of point-wise convergence) is a top. group ("$\circ$" is composition of functions).

(2) b) One can show that $\prod_{n \in \mathbb{N}} [0,1]$ equipped with the prod. topology is homogeneous, but admits no top. group structure.

(3) 13.10 Theorem:

Let G be a top. group. The sets

a) $\mathcal{U}_1 = \{ U \in \mathcal{U}(e_G) \, ; \, U = U^{-1} \}$,

b) $\mathcal{U}_2^{(n)} = \{ V^n = V \cdot \ldots \cdot V \, ; \, V \in \mathcal{U}(e_G) \}$,

c) $\mathcal{U}_3 = \{ U \in \mathcal{U}(e_G) \, ; \, U = \overline{U} \}$

are neighbourhood bases at e_G.

Proof:

a) $U \in \mathcal{U}(e_G) \overset{13.9b)}{\Longrightarrow} U^{-1} \in \mathcal{U}(e_G) \Rightarrow U \cap U^{-1} \in \mathcal{U}(e_G)$ with $U \cap U^{-1} \subseteq U$, $(U \cap U^{-1})^{-1} = U \cap U^{-1}$

b) $n=2$. $U \in \mathcal{U}(e_G) \underset{\substack{e_G \circ e_G = e_G \\ \text{"}\circ\text{" cont}}}{\Longrightarrow} \exists V \in \mathcal{U}(e_G)$: $\underset{V \cdot V}{\underbrace{\circ(V \times V)}} \subseteq U$. Then, induction on n.

c) For $U \in \mathcal{U}(e_G)$, there is $V \in \mathcal{U}(e_G)$ with $V^{-1}V \subseteq U$, since "$\circ$" and $(\cdot)^{-1}$ are continuous.

$\overset{13.9c)}{\Longrightarrow}$ $\overline{V} = \bigcap_{U \in \mathcal{U}(e_G)} UV \subseteq V^{-1}V \subseteq U$ with $\overline{V} \in \mathcal{U}(e_G)$. q.e.d.