§ 2: Convergence, Completeness and Baire's Theorem

In the following, let (X,d) be a metric space.

2.1 Definition:

A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called

a) Cauchy sequence if: $\forall \varepsilon>0 : \exists N\in\mathbb{N} : \forall n,m\geq N : d(x_n,x_m)<\varepsilon$

b) convergent to $x\in X$ if: $\forall \varepsilon>0 \; \exists N\in\mathbb{N} : \forall n\geq N : x_n\in B_\varepsilon(x)$

⌈ $\forall U\in\mathcal{U}(x)$ " " $x_n\in U$. ⌋

(we write $x_n\xrightarrow{n\to\infty}x$ or $\lim\limits_{n\to\infty}x_n=x$)

c) convergent if it converges to some $x\in X$.

(a) → **2.4. Lemma:**

For $A\subseteq X$, we have $\overline{A}=\{x\in X, \exists (x_n)_{n\in\mathbb{N}} \text{ sequence in } A : x_n\xrightarrow{n\to\infty}x\}$.

Proof

26 in [Weber], 2.3 in [Eschmeier], 2.3 in [Schweyer], 16.4 in [Fuchs]

2.5 Definition:

(X,d) is called complete if every Cauchy sequence in X converges.

2.6 Examples

a) $(\mathbb{R},d_{|\cdot|})$ is complete (Ana I),

$\mathbb{R}^n$, $\mathbb{C}^n$ are complete with $d_p\,(p\in[1,\infty))$, d_∞ (Analysis II)

b) $(\ell^\infty(M),d_M)$ is complete:

$(f_k)_{k\in\mathbb{N}}$ Cauchy sequence in $(\ell^\infty(M),d_M)$. For $x\in M$, due to

$\Rightarrow \quad |f_k(x)-f_\ell(x)|\leq \|f_k-f_\ell\|_M=d_M(f_k,f_\ell)$

for all $k,\ell\in\mathbb{N}$, $(f_k(x))_{k\in\mathbb{N}}$ is also Cauchy in $(\mathbb{C},d_{|\cdot|})$.

$\overset{(\mathbb{C},d_{|\cdot|})\text{ complete}}{\Longrightarrow}$ $\forall x\in M : (f_k(x))_{k\in\mathbb{N}}$ converges.

Let $f: M\to\mathbb{C}$, $x\mapsto \lim\limits_{k\to\infty}f_k(x)$. Let $\varepsilon>0$.

$\Rightarrow \exists N\in\mathbb{N} : \forall k,\ell\geq N : d_M(f_k,f_\ell)<\varepsilon$.

$\Rightarrow \forall x\in M.\ \forall k\geq N : |f_k(x)-f(x)|=\lim\limits_{\substack{\ell\to\infty\\ \ell\geq N}}|f_k(x)-f_\ell(x)|\leq\varepsilon$

$\Rightarrow \forall k\geq N : d_M(f_k,f)=\sup\limits_{x\in M}|f_k(x)-f(x)|\leq\varepsilon.$ $(*)$

Furthermore: $\forall x\in M.\ |f(x)|\leq|f_N(x)|+|f(x)-f_N(x)|\leq\|f_N\|_M+\varepsilon.$

$\Rightarrow \|f\|_M\leq\|f_N\|_M+\varepsilon<\infty$ so that $f\in\ell^\infty(M)$. By $(*)$, we also have $f_k\xrightarrow{k\to\infty}f$.

c) $(\mathbb{R}^{\mathbb{N}},d)$ in example 1.2 is complete.

Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{\mathbb{N}}$, $x=(x_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$. For $k\in\mathbb{N}$, write $x^{(k)}=(x_n^{(k)})_{n\in\mathbb{N}}$.

__Claim 1:__ $(x^{(k)})_{k\in\mathbb{N}}$ Cauchy in $(\mathbb{R}^{\mathbb{N}},d)\iff\forall n\in\mathbb{N}: (x_n^{(k)})_{k\in\mathbb{N}}$ Cauchy in $(\mathbb{R},d_{|\cdot|})$.

"$\Rightarrow$": Let $n\in\mathbb{N}$, $\varepsilon>0$.

$\Rightarrow \exists N\in\mathbb{N} : \forall k,\ell\geq N : \frac{1}{2^n}\frac{|x_n^{(k)}-x_n^{(\ell)}|}{1+|x_n^{(k)}-x_n^{(\ell)}|}\leq d(x^{(k)},x^{(\ell)})<\frac{1}{2^n}\frac{\varepsilon}{1+\varepsilon}$

$\overset{t\mapsto\frac{t}{1+t}\text{ increasing}}{\Longrightarrow}$ $\exists N\in\mathbb{N} : \forall k,\ell\geq N : |x_n^{(k)}-x_n^{(\ell)}|<\varepsilon$.

"$\Leftarrow$": Let $\varepsilon>0$ and $K\in\mathbb{N}$ such that $\sum\limits_{n=K}^{\infty}\frac{1}{2^n}<\frac{\varepsilon}{2}$.

Furthermore, let $N\geq K$ be such that

$$|x_n^{(k)}-x_n^{(\ell)}|<\frac{\varepsilon}{4}$$

for all $k,\ell\geq N$, $n=0,\dots,K-1$

$\Rightarrow \forall k,\ell\geq N : d(x^{(k)},x^{(\ell)})\leq\sum\limits_{n=0}^{K-1}\frac{1}{2^n}\frac{|x_n^{(k)}-x_n^{(\ell)}|}{1+|x_n^{(k)}-x_n^{(\ell)}|}+\sum\limits_{n=K}^{\infty}\frac{1}{2^n}$

$$<\left(\sum_{n=0}^{K-1}\frac{1}{2^n}\right)\varepsilon/4+\varepsilon/2<\varepsilon.$$

Similarly, we show

__Claim 2:__ $(x^{(k)})_{k\in\mathbb{N}}$ converges to $x\iff\forall n\in\mathbb{N}: x_n^{(k)}\xrightarrow{k\to\infty}x_n$ in $(\mathbb{R},d_{|\cdot|})$.

Thus, the completeness of $(\mathbb{R}^{\mathbb{N}},d)$ follows from the completeness of $(\mathbb{R},d_{|\cdot|})$.

d) Ana I: $\mathcal{C}^1[a,b]$ with d_1 from 1.2 is complete.

Inductively: $\mathcal{C}^n[a,b]$ with d_n from 1.2 is complete.

e) FA I: (ℓ^p,d_p) from 1.2 is complete for $1\leq p<\infty$

f) $X\neq\emptyset$ with the discrete metric. Cauchy sequences as well as conv. sequences are eventually constant. So, (X,d) is complete.

g) $\mathbb{Q}\subseteq\mathbb{R}$ and $\mathbb{R}\setminus\{0\}\subseteq\mathbb{R}$ as well as $\{f:\mathbb{N}\to\mathbb{C},\ f(n)=0 \text{ for fast alle } n\in\mathbb{N}\}\subseteq\ell^\infty(\mathbb{N})$ are not complete with the corresponding relative metrics.

2.7 Proposition

For all $x, x', y, y' \in X$, we have:

$$|d(x,y) - d(x',y')| \leq d(x,x') + d(y,y').$$

Proof:
Triangular inequality + ~~different~~ cases

that is, there is an isometry $j: X \hookrightarrow \tilde{X}$.

(2) → **2.9 Theorem** (completion)

Let (X,d) be a metric space.
There is a complete metric space $(\tilde{X}, \tilde{d})$ such that $X \subseteq \tilde{X}$, $\tilde{d}|_{X \times X} = d$ and $X \subseteq \tilde{X}$ is dense (that is $\overline{X}^{(\tilde{X},\tilde{d})} = \tilde{X}$).

Proof:
Define $\tilde{X} = \{(x_k)_{k\in\mathbb{N}} : (x_k)_{k\in\mathbb{N}}$ Cauchy sequence in $X\}/\sim$ with the equivalence relation:

$$(x_k)_{k\in\mathbb{N}} \sim (y_k)_{k\in\mathbb{N}} :\Leftrightarrow d(x_k, y_k) \xrightarrow{k\to\infty} 0.$$

Write $[x_k] = [(x_k)_{k\in\mathbb{N}}]$.
Consider the mapping

$$\tilde{d}: \tilde{X}\times\tilde{X} \to \mathbb{R},\ \tilde{d}([x_k],[y_k]) = \lim_{k\to\infty} d(x_k, y_k).$$

VL 2

VL 3

$\tilde{d}$ well-defined: $|d(x_k,y_k) - d(x_l,y_l)| \overset{2.7}{\leq} d(x_k,x_l) + d(y_k,y_l)$.

- $\forall k, l \in \mathbb{N}$: $|d(x_k,y_k) - d(x_l,y_l)| \leq d(x_k,x_l) + d(y_k,y_l)$.
  Since $(\mathbb{R}, d_{|\cdot|})$ is complete, $(d(x_k,y_k))_{k\in\mathbb{N}}$ converges.
- $[x_k] = [x_k'], [y_k] = [y_k']$

$$\Rightarrow |d(x_k,y_k) - d(x_k',y_k')| \leq d(x_k,x_k') + d(y_k,y_k') \xrightarrow{k\to\infty} 0$$

$$\Rightarrow \lim_{k\to\infty} d(x_k,y_k) = \lim_{k\to\infty} d(x_k',y_k').$$

$\tilde{d}$ is symmetric, positive and fulfills Δ-inequality: corresponding properties of d.
$\tilde{d}$ positively definite: def. of $\sim$
Next, consider

$$j: X \hookrightarrow \tilde{X},\ x \mapsto [(x)_{k\in\mathbb{N}}].$$

For all $x, y \in X$, we have:

$$\tilde{d}(j(x), j(y)) = \tilde{d}([x],[y]) = \lim_{k\to\infty} d(x,y) = d(x,y)$$

Then, $\tilde{d}|_{X\times X} = d$ holds.
$\Rightarrow j$ is an isometry and we can identify X with $j(X)$.
Now, let $\xi = [x_k] \in \tilde{X}$, $\varepsilon > 0$.

$\Rightarrow \exists N \in \mathbb{N}: \forall k, l \geq N: d(x_k, x_l) < \varepsilon/2$ and for $l \geq N$: $\lim_{k\to\infty} d(x_k, x_l) \leq \varepsilon/2 < \varepsilon$.
$(d(x_k,x_l))_{k\in\mathbb{N}}$ converges for all $l \in \mathbb{N}$ (2.5)

$$\Rightarrow \lim_{l\to\infty} \tilde{d}(\xi, j(x_l)) = \lim_{l\to\infty}\lim_{k\to\infty} d(x_k, x_l) = 0. \quad (*)$$

Thus, $X \subseteq \tilde{X}$ dense.
Now, let $(\xi^{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in $(\tilde{X}, \tilde{d})$, $\varepsilon > 0$.
$\Rightarrow \forall k \in \mathbb{N}: \exists x_k \in X: d(\xi^{(k)}, j(x_k)) < 1/_{k+1}$.
Choose $N \in \mathbb{N}$ such that: $d(\xi^{(k)}, \xi^{(l)}) < \varepsilon/3$, $1/_{k+1} < \varepsilon/3$ for all $k, l \geq N$.

$$\Rightarrow \forall k, l \geq N: d(x_k, x_l) \leq \tilde{d}(j(x_k), \xi^{(k)}) + \tilde{d}(\xi^{(k)}, \xi^{(l)}) + \tilde{d}(\xi^{(l)}, j(x_l))$$

$$< 1/_{k+1} + \varepsilon/3 + 1/_{l+1} < \varepsilon$$

$\Rightarrow (x_k)_{k\in\mathbb{N}}$ Cauchy sequence on (X,d). Set $\xi = [x_k] \in \tilde{X}$.

$$\Rightarrow \forall k \in \mathbb{N}: \tilde{d}(\xi^{(k)}, \xi) \leq \tilde{d}(\xi^{(k)}, j(x_k)) + \tilde{d}(j(x_k), \xi) < \frac{1}{k+1} + \tilde{d}(j(x_k), \xi) \xrightarrow[(*)]{k\to\infty} 0.$$

$\Rightarrow \lim_{k\to\infty} \xi^{(k)} = \xi$ in $(\tilde{X}, \tilde{d})$ so that $(\tilde{X}, \tilde{d})$ is complete. q.e.d.

~~**2.9 definition**~~
~~Let (X,d) be a metric space, $Y \subseteq X$. We call $d_Y = d|_{Y\times Y}$ the relative metric of d on Y.~~

2.10 Lemma

Let (X,d) be a metric space, $Y \subseteq X$. Then, we have:

(Y, d_Y) complete $\Rightarrow Y \subseteq X$ closed. If (X,d) is complete, we even have equivalence.

Proof:
"$\Rightarrow$": $(x_n)_n$ sequence on Y, $x_n \xrightarrow{n\to\infty} x \in X \overset{\text{Ex 1}}{\Rightarrow} (x_n)_n$ Cauchy in (Y, d_Y) $\Rightarrow x_n \xrightarrow{n\to\infty} y$ for some $y \in Y$

$\overset{2.2}{\Rightarrow} x = y \in \mathcal{Y}$. By 2.3, $\mathcal{Y} \subseteq X$ is closed.

"$\Leftarrow$": $(x_n)_n$ Cauchy sequence in $(\mathcal{Y}, d_{\mathcal{Y}})$

$\Rightarrow$ " " " (X, d)

$\Rightarrow x_n \xrightarrow{n\to\infty} x$ for some $x \in X$

$\underset{\text{closed}}{\overset{\mathcal{Y}\subseteq X}{\Rightarrow}} x \in \mathcal{Y} \Rightarrow x_n \xrightarrow{n\to\infty} x$ in $(\mathcal{Y}, d_{\mathcal{Y}})$. q.e.d.

2.11 Definition:

For $\{\} \neq A \subseteq X$, let

$$\operatorname{diam} A = \sup_{x,y\in A} d(x,y) \in [0,\infty]$$

be the diameter of A.

2.12 Theorem (Cantor's Intersection Theorem):

Let (X,d) be a complete metric space and let $(A_k)_{k\in\mathbb{N}}$ be a sequence of non-empty closed subsets $A_k \subseteq X$ such that $A_{k+1} \subseteq A_k$ for all $k \in \mathbb{N}$ and $\lim_{n\to\infty} \operatorname{diam} A_k = 0$. Then, there is $x \in X$ such that

$$\bigcap_{k\in\mathbb{N}} A_k = \{x\}.$$

Proof:

For $k \in \mathbb{N}$, choose $x_k \in A_k$. Due to

$$d(x_k, x_\ell) \leq \operatorname{diam}(A_{\max(k,\ell)})$$

for all $k, \ell \in \mathbb{N}$, $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in (X,d). Thus, it converges to an $x \in X$.

For all $k \in \mathbb{N}$, we have

$$x = \lim_{\substack{n\to\infty \\ n \geq k}} x_n \underset{\substack{2.4. \\ \& A_n \subseteq A_k \\ \text{for } n \geq k}}{\in} \overline{A_k} = A_k.$$

Thus, $x \in \bigcap_{k\in\mathbb{N}} A_k$.

On the other hand, let $y \in \bigcap_{k\in\mathbb{N}} A_k$.

$\Rightarrow \forall k \in \mathbb{N}: d(x,y) \leq \operatorname{diam} A_k$.

$\overset{k\to\infty}{\Rightarrow} d(x,y) = 0 \Rightarrow x = y$ q.e.d.

2.13 Theorem (Baire's Theorem)

Let (X,d) be a complete metric space, $(F_n)_{n\in\mathbb{N}}$ a sequence of closed subsets of X.

Then, the following holds:

$$\operatorname{Int}\Big(\bigcup_{n\in\mathbb{N}} F_n\Big) \neq \{\} \Rightarrow \exists n \in \mathbb{N}: \operatorname{Int}(F_n) \neq \{\}.$$

Proof:

First, we argue for the following claim:

$F \subseteq X$ closed with $\operatorname{Int}(F) = \{\} \Rightarrow \forall x_0 \in X\ \forall r > 0: \exists x_1 \in X, r_1 > 0: \overline{B}_{r_1}(x_1) \subseteq (X \setminus F) \cap B_r(x_0)$ $(*)$

This holds since $(X\setminus F) \cap B_r(x_0)$ is non-empty (due to $\operatorname{Int}(F) = \{\}$) and open.

Now, let $(F_n)_{n\in\mathbb{N}}$ be a sequence as in the assumption.

$\mathcal{A}$: We have $\operatorname{Int}(F_n) = \{\}$ for all $n \in \mathbb{N}$.

Let $x_0 \in X, r_0 > 0$ with $B_{r_0}(x_0) \subseteq \bigcup_{n\in\mathbb{N}} F_n$.

$\overset{(*)}{\Rightarrow} \exists x_1 \in X, r_1 \in (0,1): \overline{B}_{r_1}(x_1) \subseteq (X \setminus F_0) \cap B_{r_0}(x_0)$

$\overset{(*)}{\Rightarrow} \exists x_2 \in X, r_2 \in (0,\frac{1}{2}): \overline{B}_{r_2}(x_2) \subseteq (X \setminus F_1) \cap B_{r_1}(x_1)$

Set $B_1 := \overline{B}_{r_1}(x_1)$, $B_2 := \overline{B}_{r_2}(x_2)$.

Inductively, we get a sequence $(B_n)_{n\geq 1}$ of closed balls such that:

$$B_{n+1} \subseteq B_n \subseteq X \setminus F_{n-1}\ \forall n \geq 1 \text{ and } \operatorname{diam} B_n \xrightarrow{n\to\infty} 0.$$

Since X is complete, 2.12 yields:

$$\exists x \in \bigcap_{n\geq 1} B_n \subseteq \bigcap_{n\in\mathbb{N}} X \setminus F_n = X \setminus \bigcup_{n\in\mathbb{N}} F_n.$$

On the other hand, we have

$$x \in B_1 \subseteq B_{r_0}(x_0) \subseteq \bigcup_{n\in\mathbb{N}} F_n \ \lightning \quad \text{q.e.d.}$$

2.14 Definition

A set $M \subseteq X$ is called nowhere dense if $\operatorname{Int}(\overline{M}) = \{\}$.

Countable unions of nowhere dense sets are called meagre.

2.15 Corollary:

Let (X,d) be a complete metric space and $M \subseteq X$ meagre. Then, we have

$$\operatorname{Int}(M) = \{\}$$

Proof

Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of nowhere dense sets with $M=\bigcup_{n\in\mathbb{N}} M_n$.

IA: $\operatorname{Int}(\bigcup_{n\in\mathbb{N}} M_n)\neq\{\}$.

Then, $\operatorname{Int}(\bigcup_{n\in\mathbb{N}} \overline{M_n})\neq\{\} \overset{\text{Baire}}{\Rightarrow} \exists n\in\mathbb{N}: \operatorname{Int}(\overline{M_n})\neq\{\}$. $\lightning$ q.e.d.

(3) → **2.17 Corollary**:

Let (X,d) be a complete metric space and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of sets which are open and dense in X. Then, $\bigcap_{n\in\mathbb{N}} U_n \subseteq X$ is also dense.

Proof:

For $n\in\mathbb{N}$, the set

$$A_n = X\setminus U_n \subseteq X \quad \text{due to } \overline{U_n}=X$$

is closed with $\operatorname{Int}(A_n)\overset{\smile}{=}\{\}$

$\overset{\text{Baire}}{\Rightarrow} \operatorname{Int}(\bigcup_{n\in\mathbb{N}} A_n)=\{\} \Rightarrow \bigcap_{n\in\mathbb{N}} U_n = X\setminus\bigcup_{n\in\mathbb{N}} A_n \subseteq X$ is dense

q.e.d.

2.18 Remark:

Baire's Theorem doesn't hold for all metric spaces.

Consider e.g. $(\mathbb{Q}, d_{H,\mathbb{Q}})$. For all $q\in\mathbb{Q}$, the set $\{q\}\subseteq\mathbb{Q}$ is closed with $\operatorname{Int}(\{q\})=\{\}$. On the other hand, we have

$$\operatorname{Int}(\bigcup_{q\in\mathbb{Q}}\{q\}) = \operatorname{Int}(\mathbb{Q}) = \mathbb{Q}\neq\{\}.$$

(1): 2.2 Lemma

Limits in metric space are unique, that is for a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{n\to\infty} x$ and $x_n \xrightarrow{n\to\infty} y$, we have $x=y$.

Proof:

$0 \le d(x,y) \le d(x,x_n) + d(x_n,y) \overset{n\to\infty}{\Longrightarrow} d(x,y)=0 \Rightarrow x=y$ q.e.d.

2.3 Definition

Let $Y \subseteq X$. We call $d_Y = d|_{Y\times Y}$ the relative metric of d on Y.

(2) 2.8 Definition

A mapping $f: (X,d_X) \to (Y,d_Y)$ between metric spaces is called an isometry if

$$d_Y(f(x), f(y)) = d_X(x,y)$$

holds for all $x,y \in X$.

Remark:

If $f: X \hookrightarrow Y$ is an isometry we often identify X and $f(X)$ as metric spaces.
This is justified since f conserves all properties of a metric space.

(3) 2.16 Lemma

For $A \subseteq X$, we have

$$\overline{A} = X \Leftrightarrow \mathrm{Int}(X\setminus A) = \{\}$$

Proof:

"$\Rightarrow$" A: $\exists x \in \mathrm{Int}(X\setminus A)$. For a sequence $(x_n)_{n\in\mathbb{N}}$ in A with $x_n \xrightarrow{n\to\infty} x$, we have
$U \cap A \neq \{\}$ for all $U \in \mathcal{U}(x)$ ↯

"$\Leftarrow$" Let $x \in X$. For all $k \in \mathbb{N}$, there is $x_k \in B_{1/k}(x) \cap A$. Then $x_k \xrightarrow{k\to\infty} x$, so $x \in \overline{A}$.
q.e.d.