§3. Continuous functions between metric spaces

In the following, let (X,d), $(Y,\tilde{d})$ be metric spaces.

3.1 Definition:

A function $f: X \to Y$ is called

a) continuous in $x \in X$ if: $\forall \varepsilon > 0: \exists \delta > 0: f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$

b) continuous if f is continuous in every $x \in X$.

c) sequentially continuous in $x \in X$ if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{n\to\infty} x$, we have $f(x_n) \xrightarrow{n\to\infty} f(x)$.

$\ulcorner f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) \lrcorner$

d) sequentially continuous if f is sequentially continuous in every $x \in X$.

e) uniformly continuous if: $\forall \varepsilon > 0: \exists \delta > 0: \forall x \in X: f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

3.2 Theorem

For a function $f: X \to Y$, the following are equivalent

i) f is continuous,

ii) $f^{-1}(U) \subseteq X$ is open for every $U \subseteq Y$ open,

iii) $f^{-1}(A) \subseteq X$ is closed for every $A \subseteq Y$ closed,

iv) f is sequentially continuous

Proof

2.12 in [Weber], 2.13 & 2.19 in [Schreyer], 16.12 & 16.13 in [Fuchs], 2.14 & 2.23 in [Eschmeier]

3.3 Definition:

A subset $K \subseteq X$ is called compact if for every open cover $(U_i)_{i\in I}$ of K (i.e. $U_i \subseteq X$ open $\forall i \in I$, $K \subseteq \bigcup_{i\in I} U_i$) there are $i_1, \dots, i_n \in I$ such that $K \subseteq \bigcup_{j=1}^{n} U_{i_j}$.

3.4 Theorem

Let $K \subseteq X$ be equipped with the relative metric $d|_K$ and let $f: K \to Y$ be continuous.
Then, f is uniformly continuous.

Proof

Let $\varepsilon > 0$. For $x \in K$, choose $\delta_x > 0$ with $f(B^{d_K}_{\delta_x}(x)) \subseteq B_{\varepsilon/2}(f(x))$.

Then, $K \subseteq \bigcup_{x\in K} B^d_{\delta_x/2}(x)$ is an open cover

$\overset{K\subseteq X \text{ is compact}}{\Longrightarrow} \exists x_1, \dots, x_n \in K: K \subseteq \bigcup_{i=1}^{n} B^d_{\delta_i/2}(x_i)$ $\ulcorner$with $\delta_i = \delta_{x_i}$ for $i = 1,\dots,n \lrcorner$

Let $\delta := \min_{i=1}^{n} \delta_i/2$. Let $x, x' \in K$ with $d_K(x,x') < \delta$. Let $i \in \{1,\dots,n\}$ be such that $x \in B^d_{\delta_i/2}(x_i)$.

$\overset{\Delta\text{-inequality}}{\Longrightarrow} x' \in B^d_{\delta_i}(x_i)$

Thus, $\tilde{d}(f(x), f(x')) \leq \tilde{d}(f(x), f(x_i)) + \tilde{d}(f(x_i), f(x')) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$\Rightarrow$ f is uniformly continuous. q.e.d.

3.5 Theorem

Let $X_0 \subseteq X$ be dense, Y complete and $f: X_0 \to Y$ uniformly continuous.
Then, there is exactly one continuous extension $F: X \to Y$ of f (i.e. $F|_{X_0} = f$). This extension is uniformly continuous.

Proof

For every $\varepsilon > 0$, choose $\delta_\varepsilon > 0$ such that: $\forall x, y \in X_0: d(x,y) < \delta_\varepsilon \Rightarrow \tilde{d}(f(x), f(y)) < \varepsilon$.

Define

$F: X \to Y, x \mapsto \lim_{n\to\infty} f(x_n)$ if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X_0 with $x_n \xrightarrow{n\to\infty} x$.

F is well-defined

Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_0 such that $x_n \xrightarrow{n\to\infty} x$.

For $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that $d(x_n, x_m) < \delta_\varepsilon$ for all $n, m \geq N \Rightarrow \tilde{d}(f(x_n), f(x_m)) < \varepsilon$ $\forall n, m \geq N$.

$\overset{Y \text{ is complete}}{\Longrightarrow} (f(x_n))_{n\in\mathbb{N}}$ converges in $(Y, \tilde{d})$.

Now, let $(\tilde{x}_n)_{n\in\mathbb{N}}$ be another sequence in X_0 with $\tilde{x}_n \xrightarrow{n\to\infty} x$ and let $\varepsilon > 0$.

Choose $N \in \mathbb{N}$ such that

$d(x_N, x) < \frac{1}{2}\delta_{\varepsilon/3}$, $d(\tilde{x}_N, x) < \frac{1}{2}\delta_{\varepsilon/3}$, $\tilde{d}(\lim_{n\to\infty} f(x_n), f(x_N)) < \varepsilon/3$ and $\tilde{d}(\lim_{n\to\infty} f(\tilde{x}_n), \tilde{x}_N) < \varepsilon/3$.

We conclude $d(x_N, \tilde{x}_N) \leq d(x_N, x) + d(x, \tilde{x}_N) < \delta_{\varepsilon/3}$ and thus

$$\tilde{d}(\lim_{n\to\infty} f(x_n), \lim_{n\to\infty} f(\tilde{x}_n)) \leq \tilde{d}(\lim_{n\to\infty} f(x_n), f(x_N)) + \tilde{d}(f(x_N), f(\tilde{x}_N)) + d(f(\tilde{x}_N), \lim_{n\to\infty} f(\tilde{x}_n))$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(\tilde{x}_n)$.

We have $F|_{X_0} = f$ since for $x \in X_0$, we can choose the constant sequence $(x)_{n\in\mathbb{N}}$ in X_0 which converges to x.

Last, show that F is uniformly continuous.

Let $\varepsilon > 0$ and $x, y \in X$ such that $d(x,y) < \delta_{\varepsilon/2}$.

Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be sequences in X_0 such that $x_n \xrightarrow{n\to\infty} x$ and $y_n \xrightarrow{n\to\infty} y$.

Then, $\lim_{n\to\infty} d(x_n, y_n) \overset{2.7}{=} d(x,y) < \delta_{\varepsilon/2}$.

Choose $N \in \mathbb{N}$ such that

$d(x_n, y_n) < \delta_{\varepsilon/2}$

for all $n \geq N$.

We conclude for all $n \geq N$:

$$\tilde{d}(F(x), F(y)) = \tilde{d}(\lim_{n\to\infty} f(x_n), \lim_{n\to\infty} f(y_n)) \overset{2.7}{=} \lim_{n\to\infty} \tilde{d}(f(x_n), f(y_n)) \leq \varepsilon/2 < \varepsilon.$$

$\Rightarrow F$ is uniformly continuous.

F is unique as a continuous extension of f since every continuous extension is sequentially continuous.

q.e.d.