§4: Topological spaces

4.1 Definition

a) A topological space is a pair (X,τ) of a set $X^{\neq \{\}}$ and a set $\tau \subseteq \mathcal{P}(X)$ such that

i) $\{\}, X \in \tau$,

ii) $U_1, \dots, U_n \in \tau \Rightarrow \bigcap_{i=1}^{n} U_i \in \tau$,

iii) $(U_i)_{i\in I}$ family in $\tau \Rightarrow \bigcup_{i\in I} U_i \in \tau$.

The elements of τ are called open sets. τ is called a topology.
$A \subseteq X$ is called closed if $X \setminus A \subseteq X$ is open.

b) A topological space (X,τ) is called Hausdorff (or separated) if:
$\forall x,y \in X, x \neq y: \exists U, V \in \tau: x \in U, y \in V, U \cap V = \{\}$.

c) Let τ_1, τ_2 be topologies on a set X. Then, τ_2 is called finer than τ_1.
τ_1 is called coarser than τ_1 if $\tau_1 \subseteq \tau_2$

4.2 Examples

a) Let (X,d) be a metric space. By 1.6,

$$\tau = \{U \subseteq X, U \text{ is open w.r.t } d\}$$

defines a topology on X. The corresponding topological space (X,τ) is even Hausdorff:
For $x,y \in X$ with $x \neq y$, set $r = \frac{d(x,y)}{2}$. Then, $B_r(x), B_r(y) \subseteq X$ are open and disjoint with $x \in B_r(x), y \in B_r(y)$.

b) Let $X \neq \{\}$ be a set. Then $\tau = \{\{\}, X\}$ is a topology on X which is not Hausdorff for $|X| > 1$.
(indiscrete topology)

4.3 Remark

Metrics d_1, d_2 on a set X are called equivalent if there is $c > 0$ such that

$$1/c \; d_2(x,y) \leq d_1(x,y) \leq c \, d_2(x,y)$$

for all $x,y \in X$. In this case, we have

$$B^{d_1}_{\varepsilon/c}(y) \subseteq B^{d_2}_{\varepsilon}(y), \quad B^{d_2}_{\varepsilon/c}(y) \subseteq B^{d_1}_{\varepsilon}(y)$$

for all $y \in X$, $\varepsilon > 0$. Thus, equivalent metrics induce the same topology.

4.4 Definition:

Let $(X,\tau), (Y,\tilde{\tau})$ be topological spaces, $A \subseteq X, x \in X$, $f: X \to Y$ a function and $(x_n)_{n\in\mathbb{N}}$ a sequence in X.

a) $U \subseteq X$ is called neighbourhood of x if there is $V \in \tau$ such that $x \in V \subseteq U$.
We write $\mathcal{U}(x) = \{U \subseteq X;\ U \text{ neighbourhood of } x\}$.
Let

$\bar{A} = \{x \in X, \forall U \in \mathcal{U}(x): U \cap A \neq \{\}\}$ be the closure of A,

$\operatorname{Int}(A) = \{x \in A, \exists U \in \mathcal{U}(x): U \subseteq A\}$ be the interior of A and

$\partial A = \{x \in X, \forall U \in \mathcal{U}(x): U \cap A \neq \{\} \neq U \cap (X \setminus A)\}$ the boundary of A.

b) $K \subseteq X$ is called compact if for every open cover $(U_i)_{i\in I}$ of K, there are $i_1, \dots, i_n \in I$ such that $K \subseteq \bigcup_{j=1}^{n} U_{i_j}$.

c) $(x_n)_{n\in\mathbb{N}}$ converges to x if: $\forall U \in \mathcal{U}(x): \exists N \in \mathbb{N}: \forall n \geq N: x \in U$.

d) f is called continuous in x if: $\forall V \in \mathcal{U}(f(x)): \exists U \in \mathcal{U}(x): f(U) \subseteq V$.
f is called continuous if: $\forall V \subseteq Y$ open: $f^{-1}(V) \subseteq X$ is open. f is called sequentially continuous if: $\forall x \in X: \forall$ sequences $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{n\to\infty} x: f(x_n) \xrightarrow{n\to\infty} f(x)$
($\Leftrightarrow$ f is continuous in every $x \in X$)
"$\Leftarrow$": ✓ "$\Rightarrow$": Let $V \subseteq Y$ open, w.l.o.g. $f^{-1}(V) \neq \{\}$. $\forall x \in f^{-1}(V): \exists U_x \in \mathcal{U}(x)$ open: $f(U_x) \subseteq V \Rightarrow f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x \subseteq X$ open.

4.5 Lemma:

Let (X,τ) be a topological space, $A, B \subseteq X$. All the assertions from 1.10 hold.

Proof

Exactly the same as 1.10.

4.6 Remark

a) Continuous functions between topological spaces are sequentially continuous, but the reverse implication doesn't hold in general

b) For a topological space (X,τ) and $Y \subseteq X$,

$$\tau|_Y = \{U \cap Y, U \in \tau\}$$

defines a topology on Y, the relative topology.

c) Compositions of continuous mappings are continuous due to

$$f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$$

for $f: X \to Y$, $g: Y \to Z$, $V \subseteq Z$.

d) If τ_1, τ_2 are topologies on a set $X^{\neq \{\}}$, $id: (X,\tau_1) \to (X,\tau_2), x \mapsto x$ is continuous if and only if $\tau_2 \subseteq \tau_1$.
In particular, there are continuous bijections with non-continuous inverse function.

4.8 Definition

Let $X \neq \{\}$ be a set, $(F_i)_{i\in I}$ a family in $\mathcal{P}(X)$.
Then $(F_i)_{i\in I}$ has the finite intersection property (f.i.p.) if

$$\bigcap_{i\in J} F_i \neq \{\}$$

for all finite $J \subseteq I$.

4.9 Lemma:

Let (X,τ) be a topological space, $\{\} \neq Y \subseteq X$, $A \subseteq Y$, $K \subseteq X$. We have:

a) A is closed in $(Y, \tau|_Y)$ if and only if there is $B \subseteq X$ closed in (X,τ) such that

$$A = B \cap Y$$

b) A is compact in $(Y, \tau|_Y) \Leftrightarrow A$ is compact in (X,τ).

c) K is compact $\Leftrightarrow$ For every family $(F_i)_{i\in I}$ of sets closed in $(K, \tau|_K)$ with f.i.p, we have $\bigcap_{i\in I} F_i \neq \{\}$.

Proof:

a) A closed in $(Y,\tau|_Y) \Leftrightarrow \exists U \in \tau : Y\setminus A = U \cap Y$
$\Leftrightarrow \exists U \in \tau : A = Y \setminus (U \cap Y) = X \setminus U \cap Y$

b) "$\Rightarrow$" Let $(U_i)_{i\in I}$ be an open cover of A in (X,τ)
$\Rightarrow (U_i \cap Y)_{i\in I}$ open cover of A in $(Y, \tau|_Y)$
$\Rightarrow \exists i_1,\dots,i_n \in I : A \subseteq \bigcup_{j=1}^{n} (U_{i_j} \cap Y) \subseteq \bigcup_{j=1}^{n} U_{i_j}$.
"$\Leftarrow$" is very similar.

c) For a family $(U_i)_{i\in I}$ in $\mathcal{P}(X)$ and the family $(F_i)_{i\in I} = (K\setminus U_i)_{i\in I}$ of their complements, we have

- $\forall i \in I : U_i \subseteq K$ open $\Leftrightarrow \forall i \in I : F_i \subseteq K$ closed,
- $K = \bigcup_{i\in I} U_i \Leftrightarrow \bigcap_{i\in I} F_i = \{\}$,
- $\exists i_1,\dots,i_n \in I : K = \bigcup_{j=1}^{n} U_{i_j} \Leftrightarrow \exists i_1,\dots,i_n \in I : \bigcap_{j=1}^{n} F_{i_j} = \{\}$.

The equivalence follows by contrapositions. q.e.d.

4.10 Lemma

Let (X,τ), $(Y,\tilde{\tau})$ be topological spaces, $K \subseteq X$.

a) If K is compact and $A \subseteq K$ is closed in $(K, \tau|_K)$, A is compact.

b) If K is compact and (X,τ) is Hausdorff, $K \subseteq X$ is closed.

c) If K is compact and $f: X \to Y$ is continuous, $f(K)$ is compact.

Proof:

a) Let $(U_i)_{i\in I}$ be an open cover of A in $(K, \tau|_K)$.
$\Rightarrow K = K\setminus A \cup A \subseteq K\setminus A \cup \bigcup_{i\in I} U_i$
$\overset{K\setminus A \subseteq K \text{ open}}{\underset{K \text{ compact}}{\Rightarrow}} \exists i_1,\dots,i_n \in I : K \subseteq K\setminus A \cup \bigcup_{j=1}^{n} U_{i_j}$
$\Rightarrow A \subseteq \bigcup_{j=1}^{n} U_{i_j}$. Thus, A is compact.

b) We show that $X\setminus K \subseteq X$ is open. Let $x \in X\setminus K$
$\overset{X \text{ Hd}}{\Rightarrow} \forall y \in K : \exists$ open neighbourhoods $U_y \in \mathcal{U}(x)$, $V_y \in \mathcal{U}(y)$ such that $U_y \cap V_y = \{\}$.
$\Rightarrow K \subseteq \bigcup_{y\in K} V_y \overset{K \text{ comp.}}{\Rightarrow} \exists y_1,\dots,y_n \in K : K \subseteq \bigcup_{j=1}^{n} V_{y_j}$.
Then, $U_x = \bigcap_{j=1}^{n} U_{y_j} \in \mathcal{U}(x)$ is open in (X,τ) with $U_x \subseteq \bigcap_{j=1}^{n} X\setminus V_{y_j} \subseteq X\setminus K$.
In particular, $X\setminus K = \bigcup_{x\in X\setminus K} U_x \subseteq X$ is open.

c) Let $f(K) \subseteq \bigcup_{i\in I} V_i$ be an open cover in $(Y,\tilde{\tau}) \overset{f \text{ continuous}}{\Rightarrow} K \subseteq \bigcup_{i\in I} f^{-1}(V_i)$ is an open cover in (X,τ)
$\overset{K \text{ comp.}}{\Rightarrow} \exists i_1,\dots,i_n \in I : K \subseteq \bigcup_{j=1}^{n} f^{-1}(V_{i_j})$ and thus $f(K) \subseteq \bigcup_{j=1}^{n} V_{i_j}$. q.e.d

4.7 Definition

A bijective mapping $f: X \to Y$ between topological spaces (X,τ), $(Y,\tilde{\tau})$ is called a homeomorphism if f and f^{-1} are continuous.

4.11 Corollary

Let (X,τ) be a compact topological space, $(Y,\tilde{\tau})$ a Hausdorff topological space and $f: X \to Y$ be continuous.
Then, f is a homeomorphism.

Proof:

Let $A \subseteq X$ be closed $\overset{4.9a)}{\Rightarrow} A$ is compact $\overset{4.9c)}{\Rightarrow} f(A)$ is compact $\overset{4.9b)}{\Rightarrow} (f^{-1})^{-1}(A) = f(A) \subseteq Y$ is closed. q.e.d.

4.12 Remark

a) If $X \neq \{\}$ is a set and $\tau_1, \tau_2 \subseteq \mathcal{P}(X)$ are topologies such that (X, τ_1) is compact, (X, τ_2) is Hausdorff and $\tau_1 \supseteq \tau_2$, we have $\tau_1 = \tau_2$.

b) The finer a topology, the more open and closed sets, the fewer convergent series and compact sets, the smaller the closures, the bigger the interiors, the fewer continuous functions to this space and the more continuous functions on this space.

4.13 Example:

Consider $X = \mathbb{R}$ (or any uncountable set) with the topology

$$\tau = \{U \subseteq X,\ X \setminus U \text{ is at most countable}\} \cup \{\emptyset\} \quad \text{(cf. ex.)}$$

For sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{n\to\infty} x$ in (X, τ) for some $x \in X$, we have $x_n = x$ for almost every $n \in \mathbb{N}$ (else, $U = X \setminus \{x_n, x_n \neq x\}$ is open in (X, τ) with: $\forall N \in \mathbb{N} : \exists n \geq N : x_n \notin U$ ⚡)

Thus $\{x \in X, \exists (x_n)_{n\in\mathbb{N}} \text{ in } A : x_n \xrightarrow[\tau]{n\to\infty} x\} = A$ for all $A \subseteq X$. On the other hand, not every $A \subseteq X$ is closed (e.g. $A = \mathbb{N}$). Thus, $\overline{A} \neq \{x \in X, \exists (x_n)_{n\in\mathbb{N}} \text{ in } A, x_n \xrightarrow[\tau]{n\to\infty} x\}$ in general.