§5: Bases, Subbases, Countability axioms

5.1 Definition

Let (X,τ) be a topological space.

a) A subset $\mathcal{B} \subseteq \tau$ is called a basis of τ, if we have
$$U = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq U}} B$$
for every $U \in \tau$. Here, we use the convention $\bigcup_{B \in \{\}} B = \{\}$.

b) A subset $\mathcal{S} \subseteq \tau$ is called a subbasis of τ, if
$$\mathcal{B} = \{ \bigcap_{j=1}^{n} S_j ,\ n \in \mathbb{N},\ S_1, \dots, S_n \in \mathcal{S} \}$$
is a basis of τ. Here, we use the convention $\bigcap_{j=1}^{0} S_j = X$.

5.2 Examples

a) The collection of sets
$$\mathcal{S} = \{]s, \infty[,\ s \in \mathbb{R} \} \cup \{]-\infty, t[,\ t \in \mathbb{R} \}$$
is a subbasis of $\tau_{|\cdot|}$ on $\mathbb{R}$; If $U \subseteq \mathbb{R}$ is open, it can be written as a union of open intervals which can be written as intersections of elements from $\mathcal{S}$.

b) The collection of sets
$$\mathcal{B} = \{ \prod_{i=1}^{n}]a_i, b_i[;\ a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{Q} \}$$
is a (countable!) basis of $(\mathbb{R}^2, \tau_{\|\cdot\|_2})$ by

5.3 Theorem/Definition

Let $X \neq \{\}$ be a set, $\mathcal{S} \subseteq \mathcal{P}(X)$.
There is an unique topology τ on X which has $\mathcal{S}$ as a subbasis. Then, τ is the coarsest topology on X which contains $\mathcal{S}$, i.e. $\tau = \bigcap_{\substack{t \text{ top. on } X \\ \mathcal{S} \subseteq t}} t$. We call τ the topology generated by $\mathcal{S}$.

Proof:

Define
$$\mathcal{B} = \{ \bigcap_{U \in \mathcal{F}} U,\ \mathcal{F} \subseteq \mathcal{S} \text{ finite} \}$$
and
$$\tau = \{ U \subseteq X,\ U = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq U}} B \} = \{ U \subseteq X,\ U = \bigcup_{i \in I} B_i \text{ for a family } (B_i)_{i \in I} \text{ in } \mathcal{B} \} \quad (*)$$
$$= \{ U \subseteq X,\ \forall x \in U : \exists B \in \mathcal{B} : x \in B \subseteq U \}.$$

Then, $\{\}, X \in \tau$ by convention. That τ is stable under arbitrary unions follows from the second way of writing it.
By this and the fact that $\mathcal{B}$ is stable under finite intersections, we also have
$$U_1 \cap U_2 = (\bigcup_{i \in I} B_i) \cap (\bigcup_{j \in J} \tilde{B}_j) = \bigcup_{(i,j) \in I \times J} (B_i \cap \tilde{B}_j)$$
for $U_1 = \bigcup_{i \in I} B_i$, $U_2 = \bigcup_{j \in J} \tilde{B}_j \in \tau$. Thus, τ is stable under finite intersections.
Thus, τ is a topology on τ, which has $\mathcal{B}$ as a basis and thus $\mathcal{S}$ as a subbasis.
Let $\tilde{\tau}$ be another topology on X with contains $\mathcal{S}$. Then, we have $\mathcal{S} \subseteq \tilde{\tau}$ and thus $\tau \subseteq \tilde{\tau}$.
If $\mathcal{S}$ is even subbase of $\tilde{\tau}$, $\tilde{\tau} \subseteq \tau$ follows similarly. q.e.d.

5.4 Definition:

Let (X,τ) be a topological space, $x \in X$ and $\mathcal{B} \subseteq \mathfrak{U}(x)$.

a) $\mathcal{B}$ is called neighbourhood base of x if $\forall U \in \mathfrak{U}(x) : \exists B \in \mathcal{B} : x \in B \subseteq U$.

b) (X,τ) is called first-countable if every point has a countable neighbourhood base.

c) (X,τ) is called second-countable if τ has a countable base.

d) (X,τ) is called separable if there is a countable, dense subset.

→ **5.5 Lemma**:

Let (X,τ) be a topological space, $\mathcal{B}$ a basis of X. For every $x \in X$,
$$\mathcal{B}(x) := \{ B \in \mathcal{B},\ x \in B \}$$
is a neighbourhood base of x.

Proof:

For $U \in \mathfrak{U}(x)$, there is $V \in \tau$ such that $x \in V \subseteq U$. Then, by $(*)$ there is $B \in \mathcal{B}$ with $x \in B \subseteq V \subseteq U$, that is, $B \in \mathcal{B}(x)$ with $B \subseteq U$. q.e.d.

5.6 Lemma:

a) Second-countable spaces are first-countable.

b) Metric spaces are first-countable.

c) A metric space is second-countable if and only if it is separable. For topological spaces, "$\Rightarrow$" still holds.

d) (Lindelöf's Theorem) If a topological space (X,τ) is second-countable, for $\mathfrak{U} \overset{\neq \{\}}{\subseteq} \tau$, there is a sequence $(U_n)_{n \in \mathbb{N}}$ in τ such that $\bigcup_{U \in \mathfrak{U}} U = \bigcup_{n \in \mathbb{N}} U_n$.

e) If $(X,\mathcal{T})$ is a second-countable topological space and $\mathcal{B}$ a basis of $\mathcal{T}$, there is a countable basis $\mathcal{B}_0$ with $\mathcal{B}_0 \subseteq \mathcal{B}$.

Proof:

a) follows immediately from 5.5.

b) For a metric space (X,d) and $x \in X$, $\mathcal{B} = \{B_{1/n}(x), n \geq 1\}$ is a countable neighbourhood base of x.

c) "$\Rightarrow$" Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B} \subseteq \mathcal{T}$ a countable base.

For each $B \in \mathcal{B} \setminus \{\emptyset\}$, choose $x_B \in B$. Then,

$$M = \{x_B, B \in \mathcal{B} \setminus \{\emptyset\}\}$$

is countable. Now, let $x \in X$, $U \in \mathcal{U}(x)$. Let $V \subseteq X$ be open with $x \in V \subseteq U$.

Let $B \in \mathcal{B}$ with $B \subseteq V$. Then, $x_B \in M \cap U$ and thus $x \in \overline{M}$.

"$\Leftarrow$" Now, let (X,d) be a metric space and $M = \{x_n, n \in \mathbb{N}\} \subseteq X$ dense.

Let $\mathcal{B} = \{B_{1/k}(x_n), n \in \mathbb{N}, k \geq 1\}$. We show that $\mathcal{B}$ is a base of X.

Let $U^{\neq \emptyset} \subseteq X$ be open and let $x \in U$.

$\Rightarrow \exists k \geq 1 : B_{1/k}(x) \subseteq U \underset{\text{dense}}{\overset{M \subseteq X}{\Longrightarrow}} \exists n \in \mathbb{N} : x_n \in B_{1/2k}(x)$.

$\Rightarrow x \in B_{1/2k}(x_n) \underset{\Delta\text{-neq.}}{\subseteq} B_{1/k}(x) \subseteq U$.

d) Let $\mathcal{B} = \{B_k, k \in \mathbb{N}\}$ be a countable base of $(X,\mathcal{T})$.

Let $\mathcal{U}^{\neq\emptyset} \subseteq \mathcal{T}$ and define $\mathcal{J} = \{k \in \mathbb{N} : \exists U \in \mathcal{U} : B_k \subseteq U\} \subseteq \mathbb{N}$

For $k \in \mathcal{J}$, choose $U_k \in \mathcal{U}$ with $B_k \subseteq U_k$.

We show $\bigcup_{U \in \mathcal{U}} U = \bigcup_{k \in \mathbb{N}} U_k$: "$\supseteq$": ✓ "$\subseteq$": Let $U \in \mathcal{U}$, $x \in U$. There is $k \in \mathbb{N}$

with $x \in B_k \subseteq U$. Thus, $k \in \mathcal{J}$ and $x \in B_k \subseteq U_k \subseteq \bigcup_{k \in \mathbb{N}} U_k$.

e) Let $\tilde{\mathcal{B}} = \{B_k, k \in \mathbb{N}\}$ be a countable basis of $\mathcal{T}$, $\mathcal{B}$ an arbitrary basis of $\mathcal{T}$.

$\overset{\mathcal{B}\text{ basis}}{\Longrightarrow} B_k = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq B_k}} B \quad \forall k \in \mathbb{N}$

$\overset{d)}{\Longrightarrow} \forall k \in \mathbb{N} : \exists$ Folge $(B_k^{(n)})_{n \in \mathbb{N}}$ in $\mathcal{B}$: $B_k^{(n)} \subseteq B_k \ \forall n \in \mathbb{N}$, $B_k = \bigcup_{n \in \mathbb{N}} B_k^{(n)}$.

For $U \in \mathcal{T}$, we have

$$U = \bigcup_{\substack{k \in \mathbb{N} \\ B_k \subseteq U}} B_k = \bigcup_{\substack{k \in \mathbb{N} \\ B_k \subseteq U}} \bigcup_{n \in \mathbb{N}} B_k^{(n)} = \bigcup_{\substack{(k,n) \in \mathbb{N}^2 \\ B_k^{(n)} \subseteq U}} B_k^{(n)}.$$

$\Rightarrow \mathcal{B}_0 = \{B_k^{(n)}, (n,k) \in \mathbb{N}^2\} \subseteq \mathcal{B}$ is a countable base of $\mathcal{T}$.

q.e.d.

5.7 Corollary:

Let (X,d) be a metric space, $M \subseteq X$. Then:

X separable $\Rightarrow (M, d_M)$ is separable.

Proof:

X separable $\overset{5.5 c)}{\Longrightarrow}$ X has countable base $\mathcal{B}$

$\overset{\text{Ex}}{\Longrightarrow} \{B \cap M, B \in \mathcal{B}\}$ is a countable base of (M, d_M)

$\overset{5.5 c)}{\Longrightarrow} (M, d_M)$ is separable. q.e.d.

5.8 Theorem:

Let $(X,\mathcal{T})$ be a first-countable topological space.

a) For $A \subseteq X$, we have $\overline{A} = \{x \in X, \exists (x_n)_{n \in \mathbb{N}}$ sequence in $A : x_n \xrightarrow{n\to\infty} x\}$.

b) A function $f : X \to \mathcal{Y}$ to another topological space $(\mathcal{Y}, \mathcal{T}')$ is continuous if and only if it is sequentially continuous.

Proof:

a) "$\supseteq$" (holds even in general):

Let $x = \lim_{n\to\infty} x_n$ for a sequence $(x_n)_{n \in \mathbb{N}}$ in A.

Let $U \in \mathcal{U}(x)$, w.l.o.g. open $\Rightarrow \exists N \in \mathbb{N} : x_N \in U$. In particular, $U \cap A \neq \emptyset \Rightarrow x \in \overline{A}$.

"$\subseteq$" Let $x \in \overline{A}$ and $\{B_k, k \in \mathbb{N}\}$ a neighbourhood base in x.

$\Rightarrow \forall k \in \mathbb{N} : \exists x_k \in (B_0 \cap \ldots \cap B_k) \cap A$.

Let $U \in \mathcal{U}(x)$ and $N \in \mathbb{N}$ with $x \in B_N \subseteq U$.

$\Rightarrow \forall k \geq N : x_k \in B_0 \cap \ldots \cap B_k \subseteq B_N \subseteq U$. Thus, $x_k \xrightarrow{k\to\infty} x$.

b) "$\Rightarrow$" always holds by 4.6 a)

"$\Leftarrow$": Let $x \in X$, $V \in \mathcal{U}(f(x))$, $\{B_k, k \in \mathbb{N}\}$ a neighbourhood base of x.

IA: $\forall U \in \mathcal{U}(x) : f(U) \not\subseteq V$.

$\Rightarrow \forall k \in \mathbb{N} : \exists x_k \in B_0 \cap \ldots \cap B_k : f(x_k) \notin V$.

As in a), we conclude: $x_k \xrightarrow{k\to\infty} x$.

On the other hand, $V \in \mathcal{U}(f(x))$ with $f(x_k) \notin V \ \forall k \in \mathbb{N}$.

$\Rightarrow f(x_k) \overset{k\to\infty}{\not\longrightarrow} f(x)$ ↯

Thus, f is continuous in x. q.e.d.

5.9 Examples

a) $\mathbb{R}^n$ and $\mathbb{C}^n$ equipped with $\tau_{\|\cdot\|_2}$ are separable due to

$$\overline{\mathbb{Q}^n} = \mathbb{R}^n \quad \text{and} \quad \overline{\mathbb{Q}^n + i\mathbb{Q}^n} = \mathbb{C}^n.$$

b) $\mathbb{R}^{\mathbb{N}}$ is separable with respect to $d: \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, $d((x_n)_n, (y_n)_n) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n-y_n|}$:

To see that, consider

$$M = \bigcup_{n \in \mathbb{N}} \{(x_k)_{k\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}},\ x_k = 0 \text{ for all } k > n\}.$$

Let $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. For $n \geq 1$ and $k = 0, \dots, n$, choose

$$x_k^{(n)} \in \mathbb{Q} \cap \left]x_k - \tfrac{1}{n}, x_k + \tfrac{1}{n}\right[.$$

and for $n \geq 1$, let

$$x^{(n)} = (x_0^{(n)}, \dots, x_n^{(n)}, 0, 0, \dots) \in M.$$

Then, $(x^{(n)})_{n\in\mathbb{N}}$ is a sequence in M with $x^{(n)} \xrightarrow[d]{n\to\infty} x$ due to Example 2.6 c).

Thus, $\overline{M}^d = \mathbb{R}^{\mathbb{N}}$.

c) For $1 \leq p < \infty$, the space (ℓ^p, d_p) is separable with countable dense subset

$$M = \{x = (x_k)_{k\in\mathbb{N}} \in \ell^p,\ \operatorname{Re} x_k, \operatorname{Im} x_k \in \mathbb{Q}\ \forall k \in \mathbb{N},\ x_k = 0 \text{ for almost every } k \in \mathbb{N}\}.$$

d) Let $X \neq \{\}$ be equipped with the discrete metric.

Since we have $\overline{A} = A$ for all $A \subseteq X$:

$$X \text{ separable} \Leftrightarrow X \text{ countable}.$$

e) For $M \neq \{\}$, $\ell^\infty(M)$ is separable if and only if M is finite.

"$\Leftarrow$": Let $M = \{m_1, \dots, m_n\}$. Then,

$$\Phi: (\ell^\infty(M), d_M) \to (\mathbb{C}^n, d_\infty),\ f \mapsto (f(m_i))_{i=1}^n$$

is an isometric bijection $\overset{a)}{\Rightarrow} (\ell^\infty(M), d_M)$ is separable.

"$\Rightarrow$": Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in M with $x_n \neq x_m$ for $n \neq m$.

Set

$$L = \{f \in \ell^\infty(M),\ f(M) \in \{0,1\},\ f|_{M\setminus\{x_n, n\in\mathbb{N}\}} = 0\}.$$

Since

$$\mathcal{P}(\mathbb{N}) \hookrightarrow L,\ A \mapsto \left(f: M \to \mathbb{C},\ x \mapsto \begin{cases} 1, & x = x_n \text{ for an } n \in A \\ 0, & \text{else} \end{cases}\right)$$

is injective, L is uncountable.

Since $d_M|_L$ is the discrete metric, every set $L_0 \subseteq L$ is closed in $(L, d_M|_L)$

$\overset{d)}{\Rightarrow} (L, d_M|_L)$ not separable.

$\overset{5.7}{\Rightarrow} (\ell^\infty(M), d_M)$ not separable.