§6 Product topologies

6.1 Theorem

Let $X \neq \{\}$ be a set, $(X_i, \tau_i)_{i\in I}$ topological spaces and for $i \in I$, let
$$f_i : X \to X_i$$
be a function.

Let τ be the topology on X with subbase
$$\mathcal{S} = \{ f_i^{-1}(V),\ i \in I,\ V \subseteq X_i \text{ open} \} \quad (\text{exists by } 5.3)$$

Then:

a) τ is the coarsest topology on X with respect to which all the f_i are continuous,

b) A function $g : (Y, \tau') \to (X, \tau)$ on another topological space is continuous if and only if all $f_i \circ g : (Y, \tau') \to (X_i, \tau_i)$ $(i \in I)$ are continuous.

Proof.

a) It's clear that all f_i are continuous with respect to τ.
For another topology $\tilde{\tau}$ on X with respect to which all the f_i are continuous, we have $\mathcal{S} \subseteq \tilde{\tau}$ and by 5.3 also $\tau \subseteq \tilde{\tau}$.

b) "$\Rightarrow$" follows by 4.6 c)

"$\Leftarrow$": It's easy to see that
$\tilde{\tau} = \{ U \subseteq X,\ g^{-1}(U) \in \tau' \}$
is a topology on X. $\overset{f_i \circ g \text{ cont.}}{\underset{\forall i \in I}{\Longrightarrow}} \mathcal{S} \subseteq \tilde{\tau} \overset{5.3}{\Longrightarrow} \tau \subseteq \tilde{\tau} \Rightarrow g : (Y, \tau') \to (X, \tau)$ is continuous

q.e.d.

6.2 Definition

The topology in Theorem 6.1 is called the weak topology generated by the f_i $(i \in I)$.

6.3 Definition

Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces and let $X = \prod_{i\in I} X_i$.
The weak topology generated by the projections
$$\pi_i : X \to X_i,\ (x_j)_{j \in I} \mapsto x_i \ (i \in I)$$
on X is called the product topology on X.

6.4 Theorem

Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces and let $X = \prod_{i \in I} X_i$ be equipped with the product topology τ. Let (Y, τ') be another topological space. Then:

a) $\mathcal{S} = \{ \pi_i^{-1}(V),\ i \in I,\ V \in \tau_i \}$
$= \{ U = \prod_{i\in I} U_i,\ U_i \subseteq X_i \text{ open } \forall i \in I,\ \exists i_0 \in I : U_i = X_i\ \forall i \in I \setminus \{i_0\} \}$
is a subbase of the product topology and
$\mathcal{B} = \{ U = \prod_{i \in I} U_i,\ U_i \subseteq X_i \text{ open } \forall i \in I,\ \exists J \subseteq I \text{ finite} : U_i = X_i\ \forall i \in I \setminus J \}$
is a base of the product topology.

b) $f : (Y, \tau') \to (X, \tau)$ is continuous if and only if: $\forall i \in I : f_i := \pi_i \circ f : (Y, \tau') \to (X_i, \tau_i)$ continuous.

c) A sequence $(x^k)_{k\in\mathbb{N}} = ((x_i^k)_{i\in I})_{k \in \mathbb{N}}$ converges to $x = (x_i)_{i \in I} \in X$ with respect to τ if and only if

VL 6 $x_i^k \xrightarrow[\tau_i]{k\to\infty} x_i$ for all $i \in I$.

VL 7 d) Let $(X_i, d_i)_{i=1}^N$ be metric spaces. Then, the product topology on $X = \prod_{i=1}^N X_i$ is induced by the metric
$$d : X \times X \to \mathbb{R},\ ((x_i)_{i=1}^n, (y_i)_{i=1}^n) \mapsto \max_{i=1}^N d_i(x_i, y_i)$$
$$(\text{or } d : X \times X \to \mathbb{R},\ ((x_i)_{i=1}^n, (y_i)_{i=1}^n) \mapsto \sum_{i=1}^N d_i(x_i, y_i))$$

Proof:

a) follows immediately from the definitions.

b) follows from 6.1 b)

c) "$\Rightarrow$": projections are continuous by 6.1 a) and thus sequentially continuous.

"$\Leftarrow$": Let $U \in \mathcal{U}(x)$
$\overset{a) \,\&\, 5.5}{\Longrightarrow} \exists J = \{j_1, \dots, j_n\} \subseteq I$ finite, $B = \prod_{i \in I} U_i$ with $U_i \subseteq X_i$ open $\forall i \in I$, $U_i = X_i\ \forall i \in I \setminus J$:
$x \in B \subseteq U$.
Let $N \in \mathbb{N}$ with $x_{j_\ell}^n \in U_{j_\ell}\ \forall n \geq N, \ell = 1, \dots, n \Rightarrow x^n \in B \subseteq U\ \forall n \geq N$.

d) Product topology and the topology induced by d_∞ have the basis
$$\{ B_\varepsilon^d((x_i)_{i=1}^N) = \prod_{i=1}^N B_\varepsilon^{d_i}(x_i),\ (x_i)_{i=1}^N \in X,\ \varepsilon > 0 \}.$$

For the other metric(s), the assertion follows by showing that they are equivalent exactly as in Analysis II. q.e.d.

6.5 Definition:

Let (X, τ) be a topological space.

a) Let A be a **quasi-ordered** set (i.e. a set with a relation $\leq$ which is reflexive, ~~antisymmetric~~ and transitive)
Then A is called directed if
$\forall \alpha, \beta \in A : \exists \gamma \in A : \alpha \leq \gamma, \beta \leq \gamma$.

b) A net in X is a family $(x_\alpha)_{\alpha \in A}$ with a directed index set A.

For the rest of the definition, let $(x_\alpha)_{\alpha \in A}$, $(y_i)_{i \in I}$ be nets in X, $x \in X$.

c) $(x_\alpha)_{\alpha \in A}$ converges to x (write $\lim_\alpha x_\alpha = x$ or $x_\alpha \xrightarrow{\alpha} x$) if
$\forall U \in \mathfrak{U}(x) : \exists \alpha_0 \in A : \forall \alpha \geq \alpha_0 : x_\alpha \in U$

d) $(y_i)_{i \in I}$ is called a subnet of $(x_\alpha)_{\alpha \in A}$ if:
There is a function $I \to A$, $i \mapsto \alpha_i$ such that:
i) $y_i = x_{\alpha_i} \quad \forall i \in I$
ii) $i \leq j \Rightarrow \alpha_i \leq \alpha_j$
iii) $\alpha \in A \Rightarrow \exists i \in I : \alpha \leq \alpha_i$ (co-finality)

e) x is called a cluster point of $(x_\alpha)_{\alpha \in A}$ if:
$\forall U \in \mathfrak{U}(x)\ \forall \alpha \in A : \exists \beta \geq \alpha : x_\beta \in U$.

We often write $(x_{\alpha_i})_{i \in I}$ for this subnet.
Note that $|I| > |A|$ might hold!

6.6 Example:

(4) Let (X, τ) be a topological space, $x \in X$.
$\Rightarrow \mathfrak{U}(x)$ is directed by "$U \leq V :\Leftrightarrow U \supseteq V$"
$\lfloor$For $U, V \in \mathfrak{U}(x)$, $U \cap V$ is a common upper bound$\rfloor$

6.7 Theorem

Let (X, τ), (Y, τ') be topological spaces, $f : X \to Y$ a function, $x \in X$, $A \subseteq X$.

a) f is continuous in x if and only if $\forall (x_\alpha)_{\alpha \in A}$ nets in X with $x_\alpha \xrightarrow{\alpha} x : f(x_\alpha) \xrightarrow{\alpha} f(x)$.

b) $\bar{A} = \{x \in X, \exists \text{ net } (x_\alpha)_{\alpha \in A} \text{ in } A.\ x_\alpha \xrightarrow{\alpha} x\}$.

c) Let $(X_i, \tau_i)_{i \in I}$ be a family of topological spaces, $X = \prod_{i \in I} X_i$ be equipped with the product topology. Let $(x^{(\alpha)})_{\alpha \in A}$ be a net in X, $x = (x_i)_{i \in I} \in X$. For $\alpha \in A$, write $x^{(\alpha)} = (x_i^{(\alpha)})_{i \in I}$.
Then we have
$$x^{(\alpha)} \xrightarrow{\alpha} x \iff \forall i \in I : x_i^{(\alpha)} \xrightarrow{\alpha} x_i .$$

Proof:

a) "$\Rightarrow$". Let $(x_\alpha)_{\alpha \in A}$ be a net in X, $x_\alpha \xrightarrow{\alpha} x \in X$, $V \in \mathfrak{U}(f(x)) \Rightarrow \exists U \in \mathfrak{U}(x) : f(U) \subseteq V$
$\Rightarrow \exists \alpha_0 \in A : \forall \alpha \geq \alpha_0 : x_\alpha \in U$
$\Rightarrow \forall \alpha \geq \alpha_0 : f(x_\alpha) \in f(U) \subseteq V$

"$\Leftarrow$" Let f fulfill the right side.
A: $\exists V \in \mathfrak{U}(f(x)) : \forall U \in \mathfrak{U}(x) : f(U) \not\subseteq V$.
For each $U \in \mathfrak{U}(x)$, choose $x_U \in U$ with $f(x_U) \notin V$. $\Rightarrow (x_U)_{U \in \mathfrak{U}(x)}$ is a net in x.
Then: $\forall U \in \mathfrak{U}(x) : \forall \underset{V \subseteq U}{V \geq U} : x_V \in U. \Rightarrow x_U \xrightarrow{U} x$.
But $f(x_U) \notin V\ \forall U \in \mathfrak{U}(x) \Rightarrow f(x_U) \not\xrightarrow{U} f(x)$ ↯

b) "$\subseteq$". $x \in \bar{A} \Rightarrow \forall U \in \mathfrak{U}(x) : \exists x_U \in U \cap A \Rightarrow (x_\alpha)_{\alpha \in A}$ net in A with $x_\alpha \xrightarrow{\alpha} x$.
"$\supseteq$": Let $x = \lim_\alpha x_\alpha$ for a net $(x_\alpha)_{\alpha \in A}$ in A.
$\Rightarrow \forall U \in \mathfrak{U}(x) : \exists \alpha_0 \in A : x_{\alpha_0} \in U \cap A$, in particular $U \cap A \neq \{\}$ $\Rightarrow x \in \bar{A}$.

c) "$\Rightarrow$": a) & 6.1 a)
"$\Leftarrow$". Let $U \in \mathfrak{U}(x) \overset{6.4a)}{\underset{5.5}{\Longrightarrow}} \exists J = \{j_1, \ldots, j_n\} \subseteq I$ finite, $B = \prod_{i \in I} U_i$ with $U_i \subseteq X$ open $\forall i \in I$, $U_i = X\ \forall i \in I \setminus J : x \in B \subseteq U$.
$\Rightarrow \forall \nu = 1, \ldots, n : \exists \alpha_\nu \in A : \forall \alpha \geq \alpha_\nu : x_{j_\nu}^{(\alpha)} \in U_{j_\nu}$.
Inductively: $\exists \gamma \in A : \gamma \geq \alpha_\nu\ \forall \nu = 1, \ldots, n$ (A directed!)
$\Rightarrow \forall \alpha \geq \gamma : x_\alpha \in B \subseteq U$, thus $x_\alpha \xrightarrow{\alpha} x$.
q.e.d.

6.8 Theorem

Let $(X_n, \mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of metrizable topological spaces (i.e. for every $n\in\mathbb{N}$, there is a metric d_n on X_n with $\mathcal{T}_n = \mathcal{T}_{d_n}$). Then the product topology $\mathcal{T}$ on $X = \prod_{i\in I} X_i$ is metrizable by the metric $d: X\times X \to \mathbb{R}$, $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}$.

Proof:

By Example 1.2, d is a metric.

Let $(x^{(\alpha)})_{\alpha\in A}$ be a net in X, $x = (x_n)_{n\in\mathbb{N}} \in X$. For $\alpha\in A$, write $x^{(\alpha)} = (x_n^{(\alpha)})_{n\in\mathbb{N}}$.

Then, we have

$$x^{(\alpha)} \xrightarrow[\mathcal{T}]{\alpha} x \overset{6.7\,c)}{\iff} \forall n\in\mathbb{N}: x_n^{(\alpha)} \xrightarrow[\mathcal{T}_n]{\alpha} x_n \underset{\text{in the case of } \mathbb{R}^{\mathbb{N}}}{\overset{\text{just as}}{\iff}} x^{(\alpha)} \xrightarrow[\mathcal{T}_d]{\alpha} x.$$

By 6.7 a), $id: (X,\mathcal{T}) \to (X,\mathcal{T}_d)$ is a homeomorphism and thus $\mathcal{T} = \mathcal{T}_d$.

q.e.d.

(4): 6.6 Examples

a) Let $(X,\mathcal{T})$ be a topological space, $x\in X$.

$\Rightarrow \mathcal{U}(x)$ is directed by "$U \leq V \iff U \supseteq V$"

⌈For $U, V \in \mathcal{U}(x)$, $U\cap V$ is a common upper bound⌋

b) For $c\in\mathbb{R}$, the index set $A = \mathbb{R}\setminus\{c\}$ is directed by

$$x \leq y \iff |x-c| \leq |y-c|$$

For the net $(a)_{a\in A}$ in $(\mathbb{R}, \mathcal{T}_{|\cdot|})$, $x\in\mathbb{R}$, one can show

$$a \xrightarrow{a} x \iff \lim_{x\to c} x = a \quad \text{(in the sense of Analysis I)}$$

c) Let $a,b\in\mathbb{R}$ with $a<b$ and let

$$\mathcal{P} = \{((p_0,\dots,p_n),(\xi_1,\dots,\xi_n)) \mid n\in\mathbb{N},\ a\leq p_0 < \dots < p_n \leq b,\ \xi_i \in [p_{i-1},p_i] \text{ for } i=1,\dots,n\}$$

A directed quasi-ordering is given on $\mathcal{P}$ by "$((p_0,\dots,p_n),(\xi_1,\dots,\xi_n)) \leq ((q_0,\dots,q_m),(\eta_1,\dots,\eta_m)) \iff \exists (j_i)_{i=1}^n$ in $\{1,\dots,m\}$ such that: $\forall i=1,\dots,n$: $p_i = q_{j_i}$, $\xi_i = \eta_k$ for some $k\in[j_i, j_{i+1}]$"

If $f:[a,b]\to\mathbb{R}$ is Riemann-integrable, one can show

$$\lim_{\substack{P=\{p_1,\dots,p_n\}\\ \in\mathcal{P}}} \left(\sum_{i=1}^{n-1} |p_{i+1}-p_i|\, f(p_i)\right) = \int_a^b f(x)\,dx.$$

In particular, properties like additivity, monotonicity, follows from more general results.