§7. Compact sets

7.1 Definition

Let (X,τ) be a topological space, $K\subseteq X$.
K is called sequentially compact if every sequence in K has a subsequence which converges in $(K,\tau|_K)$.

7.2 Theorem

Let (X,d) be a metric space, $K\subseteq X$. Then, we have

$$K \text{ compact} \iff K \text{ sequentially compact.}$$

Proof:

"$\Rightarrow$": Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in K. For $n\in\mathbb{N}$, let

$$K_n=\overline{\{x_\ell,\ \ell\geq n\}}^K$$

$\Rightarrow (K_n)_{n\in\mathbb{N}}$ family of sets closed in $(K,\tau|_K)$ with f.i.p.

$\overset{4.9c)}{\Rightarrow} \exists x\in \bigcap_{n\in\mathbb{N}} K_n$.

$\Rightarrow \forall k\geq 1, n\in\mathbb{N}: B_{1/k}(x)\cap\{x_\ell,\ \ell\geq n\}\neq\{\}$

Inductively define a monotoneously increasing sequence $(n_k)_{k\geq1}$ in $\mathbb{N}$ with

$$x_{n_k}\in B_{1/k}(x)\quad \forall k\geq 1.$$

VL 7 $\Rightarrow (x_{n_k})_{k\geq1}$ is subsequence of $(x_n)_{n\in\mathbb{N}}$ with $x_{n_k}\xrightarrow{k\to\infty}x$.

VL 8 "$\Leftarrow$": By Ex. 25, (K,d_K) is separable.

Let $K=\bigcup_{i\in I}U_i$ be an open cover in (K,d_K)

$\overset{5.6\,d)}{\underset{7.?}{\Rightarrow}}$ $\exists$ sequence $(i_j)_{j\in\mathbb{N}}$ in I: $K=\bigcup_{j\in\mathbb{N}}U_{i_j}$.

IA: $\forall n\in\mathbb{N}\ \exists x_n\in K\setminus(U_{i_0}\cup\ldots\cup U_{i_n})$

$\overset{\text{assumption}}{\Rightarrow}$ $\exists$ conv. subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. Let $x=\lim_{k\to\infty}x_{n_k}$.

$\Rightarrow \forall n\in\mathbb{N}\quad x=\lim_{\substack{k\to\infty\\ k\geq n}}x_{n_k}\in\underbrace{K\setminus(U_{i_0}\cup\ldots\cup U_{i_n})}_{\subseteq K \text{ closed}}$ ↯ since $K=\bigcup_{j\in\mathbb{N}}U_{i_j}$.

Thus, there is a finite subcover. q.e.d.

7.3 Theorem

Let (X,τ) be a topological space, $K\subseteq X$. Then, we have

$(K,\tau|_K)$ is compact $\iff$ Every net in $(K,\tau|_K)$ has a convergent subnet (in $(K,\tau|_K)$)
$\iff$ Every net in $(K,\tau|_K)$ has a cluster point.

"+" { Proof.

"$\Rightarrow$": Let $(x_\alpha)_{\alpha\in A}$ be a net in K. For $\beta\in A$, let

$$K_\beta=\overline{\{x_\alpha,\ \alpha\geq\beta\}}^K.$$

Then, $(K_\beta)_{\beta\in A}$ is a family of closed sets in $(K,\tau|_K)$. Since A is directed, $(K_\beta)_{\beta\in A}$ has f.i.p.

$\overset{4.9c)}{\Rightarrow} \exists x\in\bigcap_{\beta\in A}K_\beta$. Let

$$I=\{(U,\alpha),\ U\in\mathcal{U}(x),x_\alpha\in U\}\neq\{\}\ (\text{since } (X,\alpha)\in I\ \forall\alpha\in A)$$

be directed by

$$(U_1,\alpha_1)\leq(U_2,\alpha_2)\ :\iff U_1\supseteq U_2 \text{ and } \alpha_1\leq\alpha_2.$$

For $(U,\alpha)\in I$, let $y_{(U,\alpha)}=x_\alpha$. Then, $(y_{(U,\alpha)})_{(U,\alpha)\in I}$ is a subnet of $(x_\alpha)_{\alpha\in I}$.
If $U_0\in\mathcal{U}(x)$, there is $\alpha_0\in A$ with $x_{\alpha_0}\in U$.

$\Rightarrow \forall (U,\alpha)\geq(U_0,\alpha_0): y_{(U,\alpha)}=x_\alpha\in U\subseteq U_0$.

Thus, $y_{(U,\alpha)}\xrightarrow{(U,\alpha)}x$.

"$\Leftarrow$": Let $\mathcal{F}\subseteq\mathcal{P}(K)$ be a family of sets closed in $(K,\tau|_K)$ with f.i.p.
By 4.9 c), it is enough to show $\bigcap_{F\in\mathcal{F}}F\neq\{\}$
Let $\mathcal{A}=\{F_1\cap\ldots\cap F_n,\ n\geq1, F_1,\ldots,F_n\in\mathcal{F}\}$ be directed by "$F\leq G:\iff F\supseteq G$"

$\overset{\text{f.i.p.}}{\Rightarrow}\forall F\in\mathcal{A}: \exists x_F\in F$. Then, $(x_F)_{F\in\mathcal{A}}$ is a net in K

$\overset{\text{assumption}}{\Rightarrow}$ $\exists$ conv. subnet $(x_{F_i})_{i\in I}$ of $(x_F)_{F\in\mathcal{A}}$. Let $x=\lim_i x_{F_i}$. Let $F_0\in\mathcal{F}\subseteq\mathcal{A}$.

$\Rightarrow \exists i_0\in I\ F_{i_0}\geq F_0$, i.e. $F_{i_0}\subseteq F$. Thus $x=\lim_{i\geq i_0}x_{F_i}\underset{F_{i_0}\subseteq K \text{ closed}}{\in}F_{i_0}\subseteq F_0$. Thus, $x\in\bigcap_{F\in\mathcal{F}}F$. Second "$\iff$" is Ex. 33e).

7.4 Remark:
Using 7.2 & 7.3 multiple times (by passing to subsequences / subnets multiple times), one can show that finite products of compact metric (or top. spaces) are compact.
Using a diagonal argument, one can show that countable products of compact metric spaces are also compact.

7.5 Definition
Let M be a partially ordered set (i.e. a non-empty set with an anti-symmetric quasi-ordering), $\mathcal{C} \subseteq M$.
a) $\mathcal{C}$ is called totally ordered or a chain if $\forall x, x' \in \mathcal{C}: x \leq x'$ or $x' \leq x$.
b) $y \in M$ is called upper bound for $\mathcal{C}$ if: $\forall x \in \mathcal{C}: x \leq y$.
c) $z \in M$ is called maximal element of M if: $x \in M, x \geq z \Rightarrow x = z$.

7.6 Remark
One can show that the following Theorem, called Zorn's Lemma holds (it is actually equivalent to the axiom of choice in ZF):
Every partially ordered set $M \neq \{\}$, in which every chain has an upper bound, has a maximal element.

7.7 Theorem (Tychonoff) ← Is actually also equivalent to the axiom of choice in ZF.
proof different! Let $(X_i, \tau_i)_{i \in I}$ be a family of compact topological spaces.
Then, $X = \prod_{i \in I} X_i$ equipped with the product topology is compact.

Proof
Let $x = (x^{(\alpha)})_{\alpha \in A}$ be a net in X.
For $J \subseteq I$, let $X_J = \prod_{i \in J} X_i$ be equipped with the product topology.
For $J \subseteq I$, we call $g \in X_J$ a partial cluster point of x if g is a cluster point of $x|_J := (x^{(\alpha)}|_J)_{\alpha \in A}$.
Let $\mathcal{P} = \{g \in X_J, J \subseteq I, g$ is a partial cluster point of $x\}$.
For $i_0 \in I$, $(x_{i_0}^{(\alpha)})_{\alpha \in A}$ has a cluster point $g_{i_0} \in X_{i_0} \hat{=} X_{\{i_0\}}$. Thus $\mathcal{P} \neq \{\}$.
Define a partial ordering on $\mathcal{P}$ by
$$(g_i)_{i \in J} \leq (h_i)_{i \in K} \Longleftrightarrow J \subseteq K \text{ and } g_i = h_i \ \forall i \in J$$
Let $\mathcal{C} = \{g^{(\lambda)} = (g_i^{(\lambda)})_{i \in J_\lambda}, \lambda \in \Lambda\}$ be a chain in $\mathcal{P}$.
Let $J = \bigcup_{\lambda \in \Lambda} J_\lambda$ and define $G = (G_i)_{i \in J} \in X_J$ by $G_i = g_i^{(\lambda)}$ if $i \in J_\lambda$.
This is well-defined since for $i \in J_\lambda$, $i \in J_\mu$, w.l.o.g $g^{(\lambda)} \leq g^{(\mu)}$ and thus $g_i^{(\lambda)} = g_i^{(\mu)}$.
Next, let $\alpha \in A$, $U \in \mathcal{U}(G)$, w.l.o.g $U = \prod_{i \in J} U_i$, $U_i \subseteq X_i$ open $\forall i \in J$, $\exists F \subseteq J$ finite: $U_i = X_i \ \forall i \in J \setminus F$.
For $f \in F$, there is $\lambda_f \in \Lambda$ with $f \in J_{\lambda_f}$
$\overset{\mathcal{C} \text{ chain}}{\Rightarrow} \exists \lambda_0 \in \Lambda: F \subseteq J_{\lambda_0}$
$\overset{g^{(\lambda_0)} \in \mathcal{P}}{\Rightarrow} \exists \beta \in A: \beta \geq \alpha$, $\forall i \in F: x_i^{(\beta)} \in U_i$, thus $x^{(\beta)} \in U$.
Thus, $G \in \mathcal{P}$ $\underset{\forall \lambda \in \Lambda: G \geq g^{(\lambda)}}{\Rightarrow}$ G is an upper bound for $\mathcal{C}$.

$\overset{\text{Zorn}}{\Longrightarrow} \exists g = (g_i)_{i \in J} \in \mathcal{P}$ maximal. IA: $J \subsetneq I$.
Let $k \in I \setminus J$. Since $g \in \mathcal{P}$, g is a cluster point of $(x^{(\alpha)}|_J)_{\alpha \in A}$.
$\overset{\text{Ex. 33e)}}{\Rightarrow} \exists$ subnet $(x^{(\alpha_t)}|_J)_{t \in T}$ of $(x^{(\alpha)}|_J)_{\alpha \in A}$ with $x^{(\alpha_t)}|_J \xrightarrow{t} g$.
(X_k, τ_k) comp. $\Rightarrow (x_k^{(\alpha_t)})_{t \in T}$ has a cluster point $p \in X_k$.
For $i \in J \cup \{k\}$, set $h_i = \begin{cases} g_i, & i \in J \\ p, & i = k \end{cases}$ and $h = (h_i)_{i \in J \cup \{k\}}$.
Let $\alpha \in A$, $U \in \mathcal{U}(h)$, w.l.o.g. $U = \prod_{i \in J \cup \{k\}} U_i$, $U_i \subseteq X_i$ open $\forall i \in J \cup \{k\}$, $\exists F \subseteq J \cup \{k\}$ finite: $U_i = X_i \ \forall i \in (J \cup \{k\}) \setminus F$
$\Rightarrow \exists t_0' \in T: \alpha_{t_0'} \geq \alpha$ and $\forall t \geq t_0': x^{(\alpha_t)}|_J \in \prod_{i \in J} U_i$.
Let $t_0 \geq t_0'$ (in part. $\alpha_{t_0} \geq \alpha_{t_0'}$) with $x_k^{(\alpha_{t_0})} \in U_k$.
$\Rightarrow \alpha_{t_0} \geq \alpha$ and $x_i^{(\alpha_{t_0})} \in U_i \ \forall i \in F$.
VL 8
Thus, $h \in \mathcal{P}$. $\underset{h \neq g}{\overset{h \geq g}{\Longrightarrow}}$ g not maximal ↯ By 7.3, our claim follows. q.e.d.

VL 9
7.8 Corollary (Heine-Borel)
Consider the top. space $(\mathbb{R}^n, \tau_{\|\cdot\|_2})$. For $K \subseteq \mathbb{R}^n$, we have:
$$K \text{ compact} \Longleftrightarrow K \subseteq \mathbb{R}^n \text{ closed + bounded.}$$

Proof:

"$\Rightarrow$": K compact $\overset{4.10b)}{\Longrightarrow}$ $K \subseteq \mathbb{R}^n$ closed.

Since $K \subseteq \bigcup_{n \in \mathbb{N}} B_n(0)$ is an open cover, K is also bounded.

"$\Leftarrow$": K bounded $\Rightarrow \exists L > 0 : K \subseteq \overline{B}_L^{d_\infty}(0) = \prod_{i=0}^{n} \overline{B}_L(0) = \prod_{i=0}^{n} [-L, L]$

By Tychonoff, $\prod_{i=0}^{n} [-L,L]$ is compact

$\overset{K \subseteq \mathbb{R}^n \text{ closed}}{\Longrightarrow}$ K compact. q.e.d.

7.9 Definition

A topological space (X, τ) is called locally compact if every $x \in X$ has a compact neighbourhood $C \in \mathcal{U}(x)$.

7.10 Examples

a) $(\mathbb{R}^n, \tau_{\|\cdot\|_2})$ and $(\mathbb{C}^n, \tau_{\|\cdot\|_2})$ are locally compact by 7.8 resp. similar arguments (for $x \in X$, $\overline{B}_{\|x\|}(0)$ is a compact neighbourhood)

b) Differentiable manifolds (even top. manifolds) are locally compact since every point has by definition a neighbourhood which is homeomorphic to an open set in some $\mathbb{R}^n$.

c) $\mathbb{R}^{\mathbb{N}}$ with the product topology is not locally compact.

Suppose, a point $x \in \mathbb{R}^{\mathbb{N}}$ had a compact neighbourhood C.

$\overset{6.4 \& 5.5}{\Longrightarrow} \exists B = \prod_{i=0}^{n} U_i \times \prod_{i=n+1}^{\infty} \mathbb{R} \subseteq C$ ($U_i \subseteq \mathbb{R}$ open $\forall i = 0, \dots, n$) with $x \in B$

$\Rightarrow \overline{B} \subseteq C$ compact $\underset{\text{continuous}}{\overset{\pi_{n+1}}{\Longrightarrow}} \pi_{n+1}(\overline{B}) = \mathbb{R}$ compact ↯

"+": d) d) By functional analysis, a normed vector space $(V, \|\cdot\|)$ is locally compact if and only if V is finite-dimensional.

7.11 Theorem (One-point-compactification)

Theorem different (same proof!)

Let (X, τ) be a Hausdorff space, $\infty \notin X$ and $\hat{X} = X \cup \{\infty\}$.

Then, $\hat{\tau} = \tau \cup \{\hat{X} \setminus K, K \subseteq X \text{ compact}\} \cup \{\hat{X}\}$

defines a topology on $\hat{X}$ such that

a) $X \subseteq \hat{X}$ is open and $\hat{\tau}|_X = \tau$,

b) $(\hat{X}, \hat{\tau})$ is a compact topological space.

If (X, τ) is locally compact, $(\hat{X}, \hat{\tau})$ is Hausdorff. If (X, τ) is not compact, $\overline{X}^{\hat{X}} = \hat{X}$ holds.

Proof:

$\hat{\tau}$ is a topology:

- $\{\}, \hat{X} \in \hat{\tau}$: ✓
- $U \in \tau$, $K, L \subseteq X$ compact $\Rightarrow \hat{X} \setminus K \cap \hat{X} \setminus L = \hat{X} \setminus (\underbrace{K \cup L}_{\substack{\text{compact} \\ \subseteq X}})$, $U \cap \hat{X} \setminus K = U \cap X \setminus K \underset{X \text{ Hd}}{\overset{4.10b)}{\in}} \tau$.

  Inductively $\hat{\tau}$ closed under finite intersections.
- For $U \in \tau$ and a family $(K_i)_{i \in I}$ of comp. subsets of X, $K \subseteq X$ compact, we have

  $\bigcup_{i \in I} (\hat{X} \setminus K_i) = \hat{X} \setminus \underbrace{\bigcap_{i \in I} K_i}_{\substack{\subseteq K_{i_0} (i_0 \in I) \\ \text{closed, thus comp.}}}$ and $U \cup \hat{X} \setminus K = \hat{X} \setminus (K \cap (X \setminus U))$

  Thus $\hat{\tau}$ is closed under unions.

Due to $X \in \tau \subseteq \hat{\tau}$, $X \subseteq \hat{X}$ is open.

Furthermore,

$$\hat{\tau}|_X = \{U \cap X, U \in \hat{\tau}\} \overset{4.10b)}{=} \tau.$$

$(\hat{X}, \hat{\tau})$ is compact:

Let $\hat{X} = \bigcup_{i \in I} U_i$ be an open cover

$\Rightarrow \exists i_0 \in I : U_{i_0} = \hat{X} \setminus K$ with $K \subseteq X$ compact, $\infty \in U_{i_0}$

Then, $K \subseteq \bigcup_{i \in I \setminus \{i_0\}} U_i \Rightarrow \exists i_1, \dots, i_n \in I \setminus \{i_0\} : \hat{X} = \hat{X} \setminus K \cup K \subseteq U_{i_0} \cup \bigcup_{j=1}^{n} U_{i_j}$

$(\hat{X}, \hat{\tau})$ is Hausdorff if (X, τ) is locally compact:

Let $x, y \in \hat{X}$, $x \neq y$, o.E. $y = \infty$ (since (X, τ) Hd. + $\tau \subseteq \hat{\tau}$)

$\Rightarrow \exists K \in \mathcal{U}(x)$ comp. in (X, τ)

$\Rightarrow K, \hat{X} \setminus K$ are disjoint neighbourhoods of x and ∞ in $(\hat{X}, \hat{\tau})$

At last, let X be not compact, $U \in \mathcal{U}(\infty)$, w.l.o.g. $U \in \hat{\tau}$ open

$\Rightarrow \exists K \subseteq X$ kp. : $U = \hat{X} \setminus K$

$\Rightarrow U \cap X = X \setminus K \neq \{\}$. q.e.d.

7.12 Theorem

Let (X, τ) be a Hausdorff space. Then:

X is locally compact $\Leftrightarrow \forall x \in X\ \forall U \in \mathcal{U}(x): \exists K \in \mathcal{U}(x)$ comp.: $K \subseteq U$.

Proof

"$\Leftarrow$": ✓

"$\Rightarrow$": Let $x \in X$, $U \in \mathcal{U}(x)$, w.l.o.g. $U \subseteq X$ open.

X loc. comp. $\Rightarrow \exists \tilde{K} \in \mathcal{U}(x)$ compact.

$\overset{4.10(a)}{\underset{4.10(b)}{\Rightarrow}}$ $A = \tilde{K} \cap (X \setminus U) \subseteq \tilde{K}$ compact

For any $y \in A$, let $U_y \in \mathcal{U}(x)$ open in X, $V_y \in \mathcal{U}(y)$ open in X: $U_y \cap V_y = \{\}$

$\Rightarrow A \subseteq \bigcup_{y \in A} V_y \overset{A\,\text{comp.}}{\Rightarrow} \exists y_1, \dots, y_n \in A: A \subseteq \bigcup_{i=1}^{n} V_{y_i}$. Let $V = \bigcup_{i=1}^{n} V_{y_i}$

$\Rightarrow U = \bigcap_{i=1}^{n} U_{y_i} \in \mathcal{U}(x)$ with $U \cap V = \{\}$.

Then $\underbrace{U \cap \mathrm{Int}(\tilde{K})}_{\neq \{\} \text{ due to } \tilde{K} \in \mathcal{U}(x)} \in \mathcal{U}(x)$ open with

$$\overline{U \cap \mathrm{Int}(\tilde{K})} \underset{4.10(b)}{\subseteq} \bar{U} \cap \tilde{K} \subseteq \underbrace{X \setminus V}_{\text{closed}} \cap \tilde{K} \subseteq X \setminus A \cap \tilde{K} = (X \setminus \tilde{K} \cup U) \cap \tilde{K} = U \cap \underbrace{\tilde{K}}_{\subseteq U} \subseteq \tilde{K}$$

$\overset{4.10(a)}{\Rightarrow} K = \overline{U \cap \mathrm{Int}(\tilde{K})}$ compact with $K \in \mathcal{U}(x)$, $K = \overline{U \cap \mathrm{Int}(\tilde{K})} \subseteq U$.

q.e.d.

7.13 Corollary:

Let (X, τ) be a locally compact Hausdorff space, $K \subseteq X$ compact, $U \subseteq X$ open with $K \subseteq U$.

$\Rightarrow \exists V \subseteq X$ open. $\bar{V}$ compact, $K \subseteq V \subseteq \bar{V} \subseteq U$.

Proof.

By 7.12: $\forall x \in K: \exists V_x \in \mathcal{U}(x)$: $V_x \subseteq X$ open, $\bar{V}_x \subseteq U$, $\bar{V}_x$ compact.

$\overset{K\,\text{comp.}}{\Rightarrow} \exists x_1, \dots, x_n \in K: K \subseteq \bigcup_{i=1}^{n} V_{x_i} =: V$

$\Rightarrow V \subseteq X$ open with $K \subseteq V$ such that $\bar{V} = \bigcup_{i=1}^{n} \bar{V}_{x_i} \subseteq U$

q.e.d.

"+"

7.14 Definition

A top. space (X, τ) is called a Baire space if for every sequence $(F_n)_{n \in \mathbb{N}}$ of sets closed in X with $\mathrm{Int}(\bigcup_{n \in \mathbb{N}} F_n) \neq \{\}$, there is $n_0 \in \mathbb{N}$ such that $\mathrm{Int}(F_{n_0}) \neq \{\}$.

7.15. Theorem

Every locally compact ~~Hd. space~~ is a Baire space. (Note: neither generalisation nor special case of 2.13)

Proof

Let $\mathrm{Int}(\bigcup_{n \in \mathbb{N}} F_n) \neq \{\}$. /A: $\forall n \in \mathbb{N}: \mathrm{Int}(F_n) = \{\}$.

~~Let~~ $x_0 \in \bigcup_{n \in \mathbb{N}} F_n$, $B_0 \subseteq X$ compact, $B_0 \in \mathcal{U}(x_0)$, $B_0 \subseteq \mathrm{Int}(\bigcup_{n \in \mathbb{N}} F_n)$

$\overset{\mathrm{Int}(F_0) = \{\}}{\Rightarrow} \mathrm{Int}(B_0) \cap X \setminus F_0 \neq \{\}$, let $x_1 \in \mathrm{Int}(B_0) \cap X \setminus F_0$, $B_1 \in \mathcal{U}(x_1)$ compact with $B_1 \subseteq \mathrm{Int}(B_0) \cap X \setminus F_0$.

$\overset{\text{inductively}}{\Rightarrow} \exists (x_n)_{n \in \mathbb{N}}$ in X, $(B_n)_{n \in \mathbb{N}}$ seq. of compact sets such that: $\forall n \in \mathbb{N}: B_n \in \mathcal{U}(x_n)$, $\forall n \geq 1: B_n \subseteq B_{n-1} \cap X \setminus F_{n-1}$

Then, $(B_n)_{n \in \mathbb{N}}$ has the f.i.p since finite intersections of its elements have non-empty interior.

$\overset{B_n \subseteq B_0}{\underset{\text{closed } \forall n \in \mathbb{N}}{\Rightarrow}} \exists x \in \bigcap_{n \in \mathbb{N}} B_n \subseteq \bigcap_{n \geq 1} B_n \subseteq \bigcap_{n \in \mathbb{N}} X \setminus F_n$. But $x \in B_0 \subseteq \bigcup_{n \in \mathbb{N}} F_n$ ↯ q.e.d