§ 8: Separation theorems

~~8.1 Definition:~~
~~Let (X,τ) be a topological space, $f: X \to \mathbb{C}$ a function. Then,~~
~~$\operatorname{supp} f = \{x \in X, f(x) \neq 0\}$~~
~~is called the support of f.~~

8.1 Definition
A topological space (X,τ) is called normal if it is Hausdorff and the following condition holds:
$\forall F, G \subseteq X$ closed, disjoint $\exists U, V \subseteq X$ open, disjoint: $F \subseteq U$, $G \subseteq V$.

(1) → **8.3 Lemma**

a) Compact Hausdorff spaces are normal.
b) Metric spaces are normal.

Proof:
a) Let (X,τ) be a comp. Hd. space, $F, G \subseteq X$ closed, disjoint.
$\overset{4.10\,a)}{\Rightarrow}$ $F, G \subseteq X$ compact, disjoint.
$\overset{\text{proof of 4.8 b)}}{\Rightarrow}$ $\forall x \in F \; \exists \underset{\text{open in } X}{U_x \in \mathcal{U}(x)}, V_x \subseteq X$ open: $G \subseteq V_x$, $U_x \cap V_x = \{\}$.
Then, $F \subseteq \bigcup_{x \in F} U_x$ is an open cover
$\Rightarrow \exists x_1, \dots, x_n \in F$: $F \subseteq \bigcup_{i=1}^{n} U_{x_i} =: U$
Let $V = \bigcap_{i=1}^{n} V_{x_i} \Rightarrow U, V \subseteq X$ open, $F \subseteq U$, $G \subseteq V$, $U \cap V = \bigcup_{i=1}^{n}\left(\left(\bigcap_{j=1}^{n} U_{x_j}\right) \cap V_{x_i}\right) = \{\}$.
b) is Exercise 15 d). q.e.d.

8.4 Theorem (Urysohn's Lemma)
Let (X,τ) be a normal topological space, $F, G \subseteq X$ be closed and disjoint. Then, there is a continuous function $f: X \to [0,1]$ such that $f|_F = 0$ and $f|_G = 1$.

Proof:
Set $U_1 = X \setminus G$. Thus, $U_1 \subseteq X$ is open with $F \subseteq U_1$.
Remark 8.2 $\Rightarrow$ $\exists U_{1/2} \subseteq X$ open: $F \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_1$
Remark 8.2 $\Rightarrow$ on $F/U_{1/2}$ resp. $\overline{U_{1/2}}/U_1$. $\exists U_{1/4}, U_{3/4} \subseteq X$ open: $F \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq U_1$.
For $n \in \mathbb{N}$, let $D_n = \{m/2^n, m = 1, \dots, 2^n\}$. Set $D = \bigcup_{n \in \mathbb{N}} D_n$ and inductively, for $d \in D$, define open sets $U_d \subseteq X$ such that: $\forall d, e \in D$: $d < e \Rightarrow F \subseteq U_d \subseteq \overline{U_d} \subseteq U_e \subseteq \overline{U_e} \subseteq U_1$.
Define
$$f: X \to [0,1],\; x \mapsto \begin{cases} 0, & x \in U_d \text{ for all } d \in D, \\ \sup\{d, x \notin U_d\}, & \text{else} \end{cases}$$
Then, surely $f|_F = 0$ and $f|_G = 1$.
Next, consider the subbase
$$\mathcal{S} = \{[0,c[, 0<c<1\} \cup \{]c,1], 0<c<1\}$$
of the topology on $[0,1]$ and let $c \in]0,1[$.
For $x \in X$, we thus have
$f(x) < c \Leftrightarrow \exists t < c: x \in U_t$
"$\Rightarrow$": def. sup
"$\Leftarrow$": $U_t \subseteq U_s \; \forall s \geq t \Rightarrow f(x) \leq t$
and
$f(x) > c \Leftrightarrow \exists t > c: x \notin \overline{U_t}$
"$\Leftarrow$": ✓
"$\Rightarrow$": $\exists s > c: x \notin U_s \Rightarrow \forall t \in]c,s[: x \notin \overline{U_t}$.
Thus,
$$f^{-1}([0,c[) = \bigcup_{\substack{t \in D \\ t < c}} U_t, \quad f^{-1}(]c,1]) = \bigcup_{\substack{t \in D \\ t > c}} X \setminus \overline{U_t}$$
are open in X
$\overset{3.22\,b)}{\Rightarrow}$ f is continuous. q.e.d.

8.5 Theorem (Tietze's Extension Theorem)
Let (X,τ) be a normal topological space, $A \subseteq X$ closed and $f: A \to [a,b]$ ($a, b \in \mathbb{R}$ with $a < b$) continuous $\Rightarrow \exists F: X \to [a,b]$ continuous: $F|_A = f$.

Proof:

w.l.o.g. $a=-1, b=1$ (since $\varphi_{a,b}:[a,b]\to[-1,1], t\mapsto \frac{2(t-a)}{b-a}-1$ is a homeomorphism)

claim 1:

For $r>0$, $h:A\to[-r,r]$ continuous, there is $H:X\to[-r/3,r/3]$ continuous with

$$\|h-H\|_A \le 2/3 r$$

Consider $A_-=h^{-1}([-r,-r/3])$, $A_+=h^{-1}([r/3,r])\subseteq X$ closed + disjoint.

Urysohn + $[-r/3,r/3]\to[0,1]$, $t\mapsto\frac{t+r/3}{2r/3}$ is a homeomorphism $\Rightarrow$ $\exists H:X\to[-r/3,r/3]$ continuous: $H|_{A_-}=-r/3$, $H|_{A_+}=r/3$.

$$\Rightarrow \|h-H\|_A\le 2r/3.$$

claim 2: There is a sequence $(g_n)_{n\ge1}$ of continuous functions $g_n:X\to\mathbb{R}$ $(n\ge1)$ such that

i) $\|g_n\|_X\le \frac{1}{3}(2/3)^{n-1}$ $\forall n\ge1$

ii) $\|f-\sum_{j=1}^n g_j\|_A\le(2/3)^n$ $\forall n\ge1$.

Induction: $n=1$: $h=f$, $r=1$ in claim 1, $g_1=H$ (different target space).

$n\to n+1$: Let $g_1,\dots,g_n$ be already defined

claim 1 with $r=(2/3)^n$, $h=(f-\sum_{j=1}^n g_j)|_A$ $\Rightarrow$ $\exists g_{n+1}:X\to[-\frac13(2/3)^n,\frac13(2/3)^n]$ continuous: $\|f-\sum_{j=1}^{n+1}g_j\|_A\le(\frac23)^{n+1}$

claim 3:

$F:X\to\mathbb{R}$, $F(x)=\sum_{n=1}^\infty g_n(x)$ is well-defined + continuous with $F(X)\subseteq[-1,1]$, $F|_A=f$.

Proof

$|g_n(x)|\le \frac13(2/3)^{n-1}$ $\forall n\ge1$ $\forall x\in X$ $\Rightarrow$ F well-defined, $|F(x)|\le\frac13\frac{1}{1-2/3}=1$ $\forall x\in X$.

Next, for all $x\in X$:

$$|F(x)-\sum_{n=1}^N g_n(x)|\le\sum_{n=N+1}^\infty|g_n(x)|\le\frac13\sum_{n=N+1}^\infty(\tfrac23)^{n-1}\xrightarrow{N\to\infty}0.$$

$$\Rightarrow \sum_{n=1}^N g_n \xrightarrow[\|\cdot\|_X]{N\to\infty} F \underset{\text{Exercise 39}}{\Rightarrow} F \text{ continuous.}$$

Furthermore

$\forall x\in A$: $|f(x)-\sum_{n=1}^N g_n(x)|\le(2/3)^N\xrightarrow{N\to\infty}0$

$\Rightarrow \forall x\in A$: $f(x)=\sum_{n=1}^\infty g_n(x)=F(x)$. Thus: $F|_A=f$. q.e.d.

8.6 Corollary:

Let (X,τ) be a normal topological space, $A\subseteq X$ closed and let $f:A\to\mathbb{R}$ be continuous.

$\Rightarrow \exists F:X\to\mathbb{R}$ continuous: $F|_A=f$.

Proof:

Consider

$g:A\to[-\pi/2,\pi/2], x\mapsto\arctan(f(x))$.

$\overset{8.5}{\Rightarrow}\exists G_0:X\to[-\pi/2,\pi/2]$ continuous: $G_0|_A=g$.

$\overset{\text{Urysohn}}{\Rightarrow}\exists\Theta:X\to[0,1]$: $\Theta|_{G_0^{-1}(\{\pm\pi/2\})}=0$, $\Theta|_A=1$.

Let $G=\Theta G_0$.

$\Rightarrow G:X\to[-\pi/2,\pi/2]$ is continuous such that $G|_A=g$, $G(X)\subseteq]-\pi/2,\pi/2[$

(2) $\to$ $\Rightarrow F:X\to\mathbb{R}$, $x\mapsto\tan(G(x))$ is continuous with $F|_A=f$. q.e.d.

8.8 Definition:

Let (X,τ) be a topological space and $(U_\alpha)_{\alpha\in A}$ an open cover of X.

A family $(\Theta_\alpha)_{\alpha\in A}$ of continuous functions $\Theta_\alpha:X\to[0,1]$ $(\alpha\in A)$ is called a partition of unity with respect to $(U_\alpha)_{\alpha\in A}$ if the following conditions hold:

i) $\operatorname{supp}\Theta_\alpha\subseteq U_\alpha$ $\forall\alpha\in A$,

ii) $(\operatorname{supp}\Theta_\alpha)_{\alpha\in A}$ is locally finite, i.e.

$\forall x\in X:\exists U\in\mathcal{U}(x):\{\alpha\in A;\operatorname{supp}\Theta_\alpha\cap U\neq\emptyset\}$ is finite.

iii) $\sum_{\alpha\in A}\Theta_\alpha(x)=1$ for all $x\in X$.

8.9 Theorem

Let X be a normal topological space and $(U_i)_{i=1}^n$ an open cover of X. Then, there is a continuous partition of unity w.r.t. $(U_i)_{i=1}^n$.

Proof:

Let $F_1 = X\setminus(U_2\cup\dots\cup U_n)\subseteq X$ closed with $F_1\subseteq U_1$.

$\overset{\text{remark}}{\underset{8.2}{\Rightarrow}}$ $\exists V_1\subseteq X$ open: $F_1\subseteq V_1\subseteq\overline{V_1}\subseteq U_1$

$\Rightarrow X = V_1\cup U_2\cup\dots\cup U_n$.

Inductively: $\exists V_1,\dots,V_n\subseteq X$ open: $X = V_1\cup\dots\cup V_n$, $\overline{V_i}\subseteq U_i$ $\forall i=1,\dots,n$.

Analogeously: $\exists W_1,\dots,W_n\subseteq X$ open: $X = W_1\cup\dots\cup W_n$, $\overline{W_i}\subseteq V_i$ $\forall i=1,\dots,n$.

Urysohn: $\exists f_1,\dots,f_n: X\to[0,1]$ continuous: $f_i|_{\overline{W_i}}=1$, $f_i|_{X\setminus V_i}=0$ $\forall i=1,\dots,n$.

In particular, supp $f_i\subseteq\overline{W_i}\subseteq U_i$ $\forall i=1,\dots,n$.

We have $\sum_{i=1}^n f_i(x)\geq 1$ for all $x\in X$. Thus, the functions

$$\Theta_i: X\to[0,1],\quad \Theta_i(x)=\frac{f_i(x)}{\sum_{j=1}^n f_j(x)}\qquad (i=1,\dots,n)$$

are as desired. q.e.d.

8.10 Remark:

If X is a normal topological space and $\mathcal{A}\subseteq C(X,[0,1])$ a subset which is closed under finite sums and quotients and which separates disjoint, closed subsets of X (i.e. $\forall F,G\subseteq X$ closed, disjoint: $\exists f\in\mathcal{A}$: $f|_G=0$, $f|_F=1$), an examination of the proof of 8.9 shows that finite open covers of normal top. spaces even admit partitions of unity with functions from $\mathcal{A}$.

8.11 Definition:

A topological space is called regular if it is Hausdorff and the following condition holds:

$\forall x\in X, F\subseteq X$ closed with $x\notin F$: $\exists U,V\subseteq X$ open, disjoint: $x\in U$, $F\subseteq V$.

8.12 Theorem

A regular top. space, which is also second-countable, is normal.
(Note that normal spaces are regular).

Proof:

Let $F,G\subseteq X$ be closed and disjoint.

Let $x\in F\Rightarrow \exists U_x\in\mathcal{U}(x)$ open, $V_x\subseteq X$ open: $V_x\supseteq G$, $U_x\cap V_x=\{\}$

$\Rightarrow \overline{U_x}\subseteq\underbrace{X\setminus V_x}_{\subseteq X\text{ closed}}\subseteq X\setminus G$, thus $\overline{U_x}\cap G=\{\}$.

Lindelöf: $\exists$ sequence $(U_n)_{n\in\mathbb{N}}$ in $\{U_x, x\in F\}$: $\bigcup_{n\in\mathbb{N}}U_n=\bigcup_{x\in F}U_x\supseteq F$.

Analogeously: $\exists$ sequence $(V_n)_{n\in\mathbb{N}}$ of sets open in X with $G=\bigcup_{n\in\mathbb{N}}V_n$ and $\overline{V_n}\cap F=\{\}$ $\forall n\in\mathbb{N}$.

For $n\in\mathbb{N}$, let

$$U_n' = U_n\cap\Big(X\setminus\bigcup_{i=0}^n\overline{V_i}\Big),\quad V_n' = V_n\cap\Big(X\setminus\bigcup_{i=0}^n\overline{U_i}\Big)\subseteq X\text{ open}$$

and

$$U=\bigcup_{n\in\mathbb{N}}U_n',\quad V=\bigcup_{n\in\mathbb{N}}V_n'\subseteq X\text{ open}.$$

Then: $U=\bigcup_{n\in\mathbb{N}}\Big(U_n\cap\underbrace{\bigcap_{i=0}^n X\setminus\overline{V_i}}_{\supseteq F\ \forall i=0,\dots,n}\Big)\supseteq F$ analogeously $V\supseteq G$.

IA: $\exists x\in U\cap V\Rightarrow\exists n,m\in\mathbb{N}: x\in U_n'\cap V_m'$, o.E. $n\leq m$

$\Rightarrow x\in U_n, x\notin U_n$ ↯. q.e.d.

8.13 Lemma:

Locally compact Hausdorff spaces are regular.

Proof:

Let (X,τ) be locally compact & Hausdorff, $(\hat{X},\hat{\tau})$ its one-point-compactification.

Let $x\in X$, $F\subseteq X$ closed with $x\notin F$

$\underset{\Rightarrow}{(\hat X,\hat\tau)\text{ Hd.}}$ $\{x\}, F\cup\{\infty\}\subseteq\hat{X}$ closed, disjoint $\ulcorner\hat{X}\setminus(F\cup\{\infty\})=X\setminus F\in\tau\subseteq\hat{\tau}\lrcorner$

$(\hat X,\hat\tau)\Rightarrow$ [illegible], thus normal: $\exists U,\hat{V}\subseteq\hat{X}$ open, disjoint: $x\in U$, $F\cup\{\infty\}\subseteq\hat{V}$

$\Rightarrow U, V=\hat{V}\cap X\subseteq X$ open, disjoint: $x\in U$, $F\subseteq V$ q.e.d.

8.14 Theorem (Urysohn's Embedding / Metrization Theorem)

Let (X,τ) be a regular space which is second-countable. Then, there is an homeomorphism $f: X\to f(X)\subseteq\mathbb{R}^{\mathbb{N}}$. In particular, (X,τ) is metrizable.

Proof:

Let $\mathcal{B}$ be a countable base for τ.

Then, $\mathcal{U} = \{(B_0, B_1) \in \mathcal{B}^2;\ \overline{B_0} \subseteq B_1\} \subseteq \mathcal{B}^2$ is countable.

Choose a surjection $\varphi : \mathbb{N} \to \mathcal{U},\ n \mapsto (B_0^{(n)}, B_1^{(n)})$.

$\overset{\text{Urysohn}}{\Longrightarrow} \forall n \in \mathbb{N} : \exists f_n : X \to [0,1]$ continuous : $f_n|_{\overline{B_0^{(n)}}} = 1,\ f_n|_{X \setminus B_1^{(n)}} = 0$.

Then, the function $f : X \to \mathbb{R}^{\mathbb{N}},\ x \mapsto (f_n(x))_{n \in \mathbb{N}}$ is continuous since all coordinate functions $\pi_n \circ f = f_n\ (n \in \mathbb{N})$ are continuous.

It remains to be shown that f is a homeomorphism on its image, i.e.

$(x_\alpha)_{\alpha \in A}$ net in X, $x \in X$: $f(x_\alpha) \xrightarrow{\alpha} f(x) \Rightarrow x_\alpha \xrightarrow{\alpha} x$.

Let $U \in \mathcal{U}(x) \overset{5.5}{\Longrightarrow} \exists B_1 \in \mathcal{B} : x \in B_1 \subseteq U$.

$\overset{X \text{ regular}}{\Longrightarrow} \exists V, W \subseteq X$ open, disjoint : $x \in V,\ X \setminus B_1 \subseteq W$.

$\overset{5.5}{\Longrightarrow} \exists B_0 \in \mathcal{B} : x \in B_0 \subseteq V \subseteq \overline{V} \subseteq \underbrace{X \setminus W}_{\subseteq X \text{ closed}} \subseteq B_1$.

Let $n \in \mathbb{N}$ with $\varphi(n) = (B_0, B_1)$.

$\lim_\alpha f_n(x_\alpha) = f_n(x) = 1 \Rightarrow \exists \alpha_0 \in A : \forall \alpha \geq \alpha_0 : f_n(x_\alpha) \neq 0$.

$\Rightarrow \forall \alpha \geq \alpha_0 : x_\alpha \in B_1 \subseteq U$. Thus: $x_\alpha \xrightarrow{\alpha} x$. q.e.d.

8.15 Lemma:

a) Subspaces and products of Hausdorff spaces are Hausdorff.

b) Subspaces and products of regular spaces are regular.

Proof:

a) Let (X, τ) be Hausdorff, $Y \subseteq X$, $x, y \in Y$ with $x \neq y$

$\Rightarrow \exists \tilde{U}, \tilde{V} \subseteq X$ open, disjoint : $x \in \tilde{U},\ y \in \tilde{V} \Rightarrow U = \tilde{U} \cap Y,\ V = \tilde{V} \cap Y \subseteq Y$ open, disjoint : $x \in U, y \in V$.

products: Ex. 30 a).

b) Let (X, τ) be regular, $Y \subseteq X$, $F \subseteq Y$ closed, $x \in Y \setminus F$.

$\overset{4.9a)}{\Longrightarrow} \exists G \subseteq X$ closed : $F = G \cap Y$

$\overset{X \text{ regular}}{\Longrightarrow} \exists \tilde{U}, \tilde{V} \subseteq X$ open, disjoint : $x \in \tilde{U},\ G \subseteq \tilde{V}$

$\Rightarrow U = \tilde{U} \cap Y,\ V = \tilde{V} \cap Y \subseteq Y$ open, disjoint : $x \in U,\ F = G \cap Y \subseteq V$.

Next, let $(X_i, \tau_i)_{i \in I}$ be a family of regular top. spaces, $X = \prod_{i \in I} X_i$ equipped with the product topology.

Let $F \subseteq X$ be closed, $x = (x_i)_{i \in I} \in X \setminus F$.

$\overset{6.4 \& 5.5}{\Longrightarrow} \exists J = \{j_1, \dots, j_n\} \subseteq I$ finite, $B = \prod_{i \in I} U_i$ such that $U_i \subseteq X_i$ open $\forall i \in I$, $U_i = X_i\ \forall i \in I \setminus J$:

$x \in B \subseteq X \setminus F$.

$\Rightarrow \forall k = 1, \dots, n : \exists V_{j_k} \subseteq X_{j_k}$ open : $x_{j_k} \in V_{j_k} \subseteq \overline{V}_{j_k} \subseteq X \setminus (X \setminus U_{j_k}) = U_{j_k}$.

Let

$$V = X \setminus (\pi_{j_1}^{-1}(\overline{V}_{j_1}) \cap \dots \cap \pi_{j_n}^{-1}(\overline{V}_{j_n})) \subseteq X, \quad U = \bigcap_{k=1}^{n} \pi_{j_k}^{-1}(V_{j_k}) \subseteq X \text{ open.}$$

Then,

$$F \subseteq X \setminus B = X \setminus \bigcap_{k=1}^{n} \pi_{j_k}^{-1}(U_{j_k}) \subseteq V,\ x \in U \text{ and } U \cap V = \{\}.$$

q.e.d.

8.16 Remark:

Subspaces and (even finite) products of normal spaces need not be normal again, cf. Munkres, Topology - a first course, Ex. 2 in §4.2.

8.17 Corollary:

For a topological space (X, τ), the following are equivalent:

i) X is metrizable and second-countable,

ii) X is metrizable and separable,

iii) X is regular and second-countable,

iv) X is homeomorphic to a subspace of $\mathbb{R}^{\mathbb{N}}$.

Proof

i) $\Leftrightarrow$ ii) : 5.6 c)

ii) $\Rightarrow$ iii) : 8.3 & 5.6 c)

iii) $\Rightarrow$ iv) : 8.14

iv) $\Rightarrow$ i) : 6.8 : $\mathbb{R}^{\mathbb{N}}$ metrizable

5.9 b): $\mathbb{R}^{\mathbb{N}}$ separable $\overset{5.7}{\Longrightarrow}$ X separable $\overset{5.6 c)}{\Longrightarrow}$ X second-countable q.e.d.

8.18 Remark

There are several other notions of separability:

top. space $\overset{\Leftarrow}{\not\Rightarrow}$ Kolmogorov or T0 space $\overset{\Leftarrow}{\not\Rightarrow}$ Fréchet or T1 space $\overset{\Leftarrow}{\not\Rightarrow}$ Hausdorff or T2 space $\overset{\Leftarrow}{\not\Rightarrow}$ regular: T3 + T1 $\overset{\not\Rightarrow}{\Leftarrow}$ completely regular $\overset{\not\Rightarrow}{\Leftarrow}$ normal: T4 + 1

T2 space $\not\Leftarrow$ T3; T3 $\not\Leftarrow$ T4 ...

top. space – T0: indiscrete top.

T0: $\forall x, y \in X, x \neq y$: $\exists U \in \tau$: $x \in U$, $y \notin U$

T0 – T1: $X = \{0,1\}$, …

T1: $\forall x, y \in X, x \neq y$: $\exists U, V \in \tau$: $x \in U, y \notin U$, $x \notin V, y \in V$

T1 – T2: cofinite topology, Zariski topology

T2 – regular: top. on $\mathbb{Z}$ from ex. 24

T2 – T3: $X = \{0,1\}$, $\tau = \{\{\}, \{0\}, X\}$

T3: closed sets + points sep. by open sets

completely regular: T_1 & closed sets + points sep. by cont. functions

T4: closed sets sep. by open sets

Some authors use T3/reg. respectively T4/normal the other way around.

(1): 8.2 Remark:

Let X be a normal top. space and $F \subseteq X$ closed, $W \subseteq X$ open with $F \subseteq W$.

$\Rightarrow \exists U, V \subseteq X$ open, disjoint: $F \subseteq U$, $X \setminus W \subseteq V$. We conclude:

$$F \subseteq U \subseteq \bar{U} \subseteq \underbrace{X \setminus V}_{\subseteq X \text{ closed}} \subseteq W$$

(2): 8.7 Definition:

Let (X, τ) be a top. space and $f: X \to \mathbb{C}$ a function. Then,

$$\operatorname{supp} f = \overline{\{x \in X, f(x) \neq 0\}}$$

is called the support of f.