§9: Connectedness

9.1 Definition:
Let (X,τ) be a top. space.
a) X is called connected if $\forall U,V\subseteq X$ open, non-empty: $X=U\cup V \Rightarrow U\cap V\neq \{\}$.
b) X is called path-connected if: $\forall x,y\in X: \exists \gamma: [a,b]\to X$ continuous: $\gamma(a)=x, \gamma(b)=y$.

9.2 Lemma:
Let (X,τ), (Y,t) be topological spaces and $f: X\to Y$ continuous. Then, we have
a) If X is (path-) connected, so is $f(X)$.
b) If X is path-connected, X is connected.
Proof:
Exercise

9.3 Lemma:
Let (X,τ) be a topological space. Then, we have:
X is not connected $\Leftrightarrow \exists f: X\to (\{0,1\}$, discrete topology$)$ surjective + continuous.
Proof:
"$\Leftarrow$" follows from $X=f^{-1}(\{0\})\cup f^{-1}(\{1\})$.
"$\Rightarrow$": Let $U,V\subseteq X$ be open, disjoint with $X=U\cup V$, $U\neq\{\}\neq V$.
$\Rightarrow f: X\to \{0,1\}$; $x\mapsto \begin{cases}0, x\in U\\ 1, x\in V\end{cases}$ is surjective and continuous since $f^{-1}(\{0\})=U$, $f^{-1}(\{1\})=V$.
q.e.d.

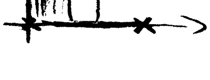
9.4 Lemma:
Let (X,τ) be a top. space and $(X_i)_{i\in I}$ a family of subsets of X such that $X_i\cap X_j\neq\{\}$ for all $i,j\in I$.
If all $(X_i, \tau|_{X_i})$ are (path-) connected, so is $\bigcup_{i\in I} X_i$.
Proof:
First, let all X_i be path-connected, $x\in X_i$, $y\in X_j$ $(i,j\in I)$ and $p\in X_i\cap X_j$.
$\Rightarrow \exists \gamma_1: [0,1]\to X_i$, $\gamma_2: [1,2]\to X_j$ continuous with $\gamma_1(0)=x$, $\gamma_1(1)=p=\gamma_2(1)$, $\gamma_2(2)=y$.
Let
$$\gamma: [0,2]\to \bigcup_{i\in I} X_i,\ \gamma(t)=\begin{cases}\gamma_1(t), & t\in[0,1]\\ \gamma_2(t), & t\in[1,2]\end{cases}$$
Then, γ is continuous (use similar result to Ex. 17) with $\gamma(0)=x$, $\gamma(1)=y$.
Now, let all X_i be connected and $f: \bigcup_{i\in I} X_i\to\{0,1\}$ continuous.
Let $i_0\in I$. Then, $f(X_{i_0})\subseteq\{0,1\}$ is connected, w.l.o.g. $f(X_{i_0})=\{0\}$. If $j\in I$ is another index,
we have $X_{i_0}\cap X_j\neq\{\}$, thus $0\in f(X_j)$.
$\xRightarrow[\text{connected}]{f(X_j)\subseteq\{0,1\}} f(X_j)=\{0\}$. Thus, f is constant. By 9.3, $\bigcup_{i\in I} X_i$ is connected.
q.e.d.

9.5 Lemma:
Let (X,τ) be a top. space, $A\subseteq X$ connected, $B\subseteq X$ with $A\subseteq B\subseteq \bar{A}$.
Then, B is also connected. In particular, $\bar{A}$ is connected.
Proof
Let $f: B\to\{0,1\}$ be continuous $\overset{9.3}{\Rightarrow} f|_A$ is constant
$\xRightarrow[\text{dense}]{A\subseteq B}$ f constant $\overset{9.3}{\Rightarrow}$ B connected.
q.e.d.

9.6 Lemma/Definition
Let (X,τ) be a top. space, $x,y\in X$. We write
$x\sim y :\Leftrightarrow \exists A\subseteq X$ connected: $x,y\in A$
and $x \overset{p}{\sim} y :\Leftrightarrow \exists A\subseteq X$ path-connected: $x,y\in A$.
This defines equivalence relations $\sim$ and $\overset{p}{\sim}$ on X.

9.7 Definition
The equivalence classes w.r.t. $\sim$ resp. $\overset{p}{\sim}$ are called connected components resp.
path-connected components of X.

9.8 Lemma:
For a top. space (X,τ), we have:
a) X is the disjoint union of its connected components.
b) Every connected subspace $M\subseteq X$ is included in a connected component of X.
c) The connected components of X are connected
d) The connected components of X are closed.
Apart from d), everything stays true for path-connectedness.
Proof:
a) is true for every equivalence relation.
b) Let $M\subseteq X$ be connected, $p\in M \xRightarrow[\forall x\in M]{x\sim p} M\subseteq [p]_\sim$.
c) Let $C\subseteq X$ be a connected component of X, $p\in C$.
$\overset{9.4}{\Rightarrow} C \overset{a),b)}{=} \bigcup_{\substack{M\subseteq X \text{ connected}\\ p\in M}} M$ connected.
d) Let $C\subseteq X$ be a connected component of $X \overset{c)\,\&\,9.5}{\Rightarrow} \bar{C}\subseteq X$ connected $\overset{a),b)}{\Rightarrow} \bar{C}=C$. q.e.d.

9.9 Example:
Let $K=\{\frac{1}{n}, n\geq 1\}$, $p=(0,1)\in\mathbb{R}^2$ and
$$X=([0,1]\times\{0\})\cup(K\times[0,1])\cup\{p\}$$

be equipped with the relative topology of $\mathbb{R}^2$.
Since
$$A=([0,1]\times\{0\})\cup(K\times[0,1])\subseteq\mathbb{R}^2$$
is path-connected, it is also connected by 9.2b).
$\overset{9.5}{\Rightarrow}$ X connected. We will show that X is not path-connected.

Let $\gamma : [a,b] \to X$ be continuous with $\gamma(a) = p$.

$\Rightarrow \gamma^{-1}(\{p\}) \subseteq [a,b]$ is closed.

Let $x_0 \in \gamma^{-1}(\{p\}) \Rightarrow \exists \varepsilon > 0 : U = [a,b] \cap]x_0 - \varepsilon, x_0 + \varepsilon[\subseteq \gamma^{-1}(B_1(p))$

A: $\exists q \in \gamma(U) \setminus \{p\} \Rightarrow \exists n \geq 1, t \in [0,1] : q = (1/n, t)$. Let $r \in]\frac{1}{n+1}, \frac{1}{n}[$

$\Rightarrow \gamma(U) \subseteq (]-\infty, r[\times \mathbb{R}) \cup (]r, \infty[\times \mathbb{R})$ with $q \in (]r,\infty[\times \mathbb{R}) \cap \gamma(U)$, $p \in (]-\infty, r[\times \mathbb{R}) \cap \gamma(U)$ $\lightning$.

since $\gamma(U)$ is connected by 9.2 $\Rightarrow \gamma(U) = \{p\}$

$\Rightarrow \gamma^{-1}(\{p\}) = U \subseteq [a,b]$ closed & open

$\overset{[a,b] \text{ connected}}{\Rightarrow} [a,b] \setminus \gamma^{-1}(\{p\}) = \{\} \Rightarrow \gamma \equiv p$. Thus, X is not path-connected.

9.10 Definition:

Let (X, τ) be a top. space. Then, X is called locally (path-)connected if:

$\forall x \in X : \forall U \in \mathcal{U}(x) : \exists C \in \mathcal{U}(x)$ (path-)connected: $C \subseteq U$.

⌜loc. (path)-connected $\not\Rightarrow$ (path)-connected: $\mathbb{R} \setminus \{0\}$
(path)-connected $\not\Rightarrow$ locally (path)-connected: cf. Ex. 9.9⌟

9.11 Remark:

Since balls w.r.t. norms are convex and thus path-connected, normed spaces are locally path-connected and thus locally connected. ⌜In fact, even top. vector spaces are even path-connected⌟

9.12 Theorem:

A top. space (X, τ) is locally (path-)connected if and only if

$\forall U \subseteq X$ open: $\forall$ (path-)connected components C of U: $C \subseteq X$ open.

Proof:

"$\Rightarrow$": Let $U \subseteq X$ be open, $C \subseteq U$ a (path-)connected component, $x \in C$.

$\Rightarrow \exists V \in \mathcal{U}(x)$ (path-)connected: $V \subseteq U \overset{9.8b),a)}{\Rightarrow} V \subseteq C$. Thus, $C \subseteq X$ is open.

"$\Leftarrow$": Let $x \in X, U \in \mathcal{U}(x) \Rightarrow \exists V \in \mathcal{U}(x), V \subseteq X$ open: $V \subseteq U$.

Let $C \subseteq V$ be the (path-)connected component of x in V

$\overset{\text{assumption}}{\Rightarrow} C \subseteq X$ open, thus $C \in \mathcal{U}(x)$. By 9.8 c), C is connected. q.e.d.

9.13 Theorem:

Let (X, τ) be locally path-connected, $U \subseteq X$ open.

a) U is connected if and only if U is path-connected.

b) The connected components of U are the path-connected components of U.

Proof:

a) "$\Leftarrow$": 9.2

"$\Rightarrow$": Let U be connected, $p \in U$, $C \subseteq U$ the path-connected component of p in U.

$\overset{9.12}{\Rightarrow} C \subseteq X$ open.

A: $\exists a \in U \setminus C \Rightarrow \exists W \in \mathcal{U}(a)$ path-connected: $W \subseteq U$.

If $W \cap C \neq \{\}$, $W \cup C$ were path-connected by 9.4, 9.8c) with $W \cup C \supsetneq C$ $\lightning$

Thus, $W \subseteq U \setminus C$.

$\Rightarrow C, U \setminus C \subseteq X$ open $\Rightarrow C, U \setminus C \subseteq U$ open, $p \in C, a \in U \setminus C$ $\lightning$ (U connected)

Thus, $U = C$ is path-connected by 9.8a).

b) Let $C \subseteq U$ be a connected component, $p \in U$ with $C = [p]_\sim$. By 9.12, $C \subseteq X$ is open, thus by 9.8c) & 9.13 a), it is path-connected $\overset{9.8b),a)}{\Rightarrow} C \subseteq [p]_\approx$

The same argument shows $[p]_\approx \subseteq [p]_\sim$. q.e.d.