Topology

held by M. Sc. Sebastian Langendörfer in winter 2017

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General organisation and information

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1 Metric spaces

Definition 1.1 (Metric space): Let $X \neq \{\}$. A function $d : X \times X \to \mathbb{R}$ is called a *metric*, if the following conditions hold for all $x, y, z \in X$:

(i) $d(x, y) \geq 0$,

(ii) $d(x, y) = 0 \Leftrightarrow x = y$, (positive definiteness)

(iii) $d(x, y) = d(y, x)$, (symmetry)

(iv) $d(x, z) \leq d(x, y) + d(y, z)$. (triangular inequality)

(X, d) is then called a metric space.

Example 1.2: (i) $X = \mathbb{R}^n$ or $X = \mathbb{C}^n$ can be equipped with the following metrics: For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in X$, put

$$d_1(x,y) := \sum_{i=1}^n |x_i - y_i| \quad d_p(x,y) := \Big(\sum_{i=1}^n |x_i - y_i|^p\Big)^{\frac{1}{p}} \quad d_\infty(x,y) := \max_{1 \leq i \leq n} |x_i - y_i|$$

(ii) Let $(E, \|\cdot\|)$ be a normed space. Then $\|\cdot\|$ induces a metric

$$d_{\|\cdot\|} : E \times E \to \mathbb{R} \quad , \quad d(x,y) := \|x - y\|.$$

(iii) Let $M \neq \{\}$. For $f : M \to \mathbb{C}$, let $\|f\|_M = \sup_{x \in M} |f(x)| \in [0, \infty]$. Set

$$X = \ell^\infty(M) = \{f : M \to \mathbb{C} \mid \|f\|_M < \infty\}.$$

One can check, that $\|\cdot\|_M$ is a norm and thus induces a metric $d_{\|\cdot\|_M}$.

(iv) Let $X = \mathbb{R}^{\mathbb{N}} = \{(x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{R} \,\forall n \in \mathbb{N}\}$ – this construction works analogeously with $\mathbb{C}^{\mathbb{N}}$ – and

$$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R} \\ ((x_n), (y_n)) &\longmapsto \sum_{n=0}^\infty \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}. \end{aligned}$$

This map is welldefined via the dominated convergence theorem. Positive definiteness and symmetry are clear. For the triangular inequality we note that

$$\begin{aligned} f : (-1, \infty) &\longrightarrow \mathbb{R} \\ t &\longmapsto \frac{t}{1+t} \end{aligned}$$

is strictly increasing because $f'(t) = \frac{1}{(1+t)^2} > 0$ for all $t \in (-1, \infty)$. Thus, for $\alpha, \beta, \gamma \geq 0$ with $\alpha \leq \beta + \gamma$, we have

$$\frac{\alpha}{1+\alpha} \leq \frac{\beta+\gamma}{1+\beta+\gamma} \leq \frac{\beta}{1+\beta} + \frac{\gamma}{1+\gamma}.$$

For $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}}$ and $z = (z_n)_{n\in\mathbb{N}}$ we conclude

$$\begin{aligned} d(x,y) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n + y_n|} \\ &\le \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|x_n - z_n|}{1 + |x_n - z_n|} + \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|z_n - y_n|}{1 + |z_n - y_n|} = d(x,z) + d(z,y). \end{aligned}$$

(v) Set $X = C^n[a,b] = \{f : [a,b] \to \mathbb{C} : f \text{ is } n\text{-times continuously differentiable}\}$ for a perfect interval $[a,b]$. Then

$$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R} \\ d(f,g) &\longmapsto \max_{0\le i\le n} \|f^{(i)} - g^{(i)}\|_{[a,b]} \end{aligned}$$

declares a metric on X.

(vi) For $1 \le p < \infty$, let $\ell^p = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{n=0}^{\infty} |x_n|^p < \infty\}$. Then we define a metric on ℓ^p via

$$\begin{aligned} d : \ell^p \times \ell^p &\longrightarrow \mathbb{R} \\ ((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) &\longmapsto \Big(\sum_{n=0}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}}. \end{aligned}$$

In functional analysis 1, we show that this is indeed a metric space.

(vii) Let $X \neq \{\}$ and define

$$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}. \end{aligned}$$

Then d is a metric on X, the so-called discrete metric.

In the following text, let (X,d) be a metric space.

Definition 1.3: Let $a \in X$, $r > 0$.

(i) We call $B_r(a) = \{x \in X : d(x,a) < r\}$ the *open ball* with radius r in a and $\overline{B}_r(a) = \{x \in X : d(x,a) \le r\}$ the *closed ball* with radius r in a.

(ii) A set $U \subset X$ is called open (in X), if the following holds:

$$\forall\, x \in U \,\exists\, \varepsilon > 0 : B_\varepsilon(x) \subseteq U.$$

A set $F \subseteq X$ is closed (in X), if $X \setminus F \subset X$ is open.

(iii) $U \subseteq X$ is called a *neighbourhood* of a, if there is $V \subset X$ open such that $a \in V \subset U$.

(iv) $\mathfrak{U}(a) = \{U \subseteq X : U \text{ is a neighbourhood of } a\}$.

Proposition 1.4: (i) *Open balls are open,*

(ii) *Closed balls are closed.*

Proof: (i) Let $B_r(a) \subseteq X$ and $x \in B_r(a)$. Put $\varepsilon = r - d(x,a) > 0$. For all $y \in B_\varepsilon(x)$ it holds that

$$d(y,a) \leq d(y,x) + d(x,a) < r - d(x,a) + d(x,a) = r,$$

so $B_\varepsilon(x) \subseteq B_r(a)$ and thus $B_r(a) \subseteq X$ is open.

(ii) Let $\overline{B}_r(a) \subseteq X$ and $x \in X \setminus \overline{B}_r(a)$. Put $\varepsilon = d(x,a) - r > 0$. For $y \in B_\varepsilon(x)$ we get

$$d(y,a) \geq d(x,a) - d(x,y) > d(x,a) - (d(x,a) - r) = r,$$

so $B_\varepsilon(x) \subseteq X \setminus \overline{B}_r(a)$ and $\overline{B}_r(a) \subseteq X$ is closed. ∎

Remark 1.5: Let $X \neq \{\}$ be equipped with the discrete metric, $U \subseteq X$ and $x \in U$. Then, we have $B_{\frac{1}{2}}(x) = \{x\}$, so all sets are open and all sets are closed.

Lemma 1.6: (i) $\{\}$ *and* X *are open,*

(ii) *If* $U_1, \ldots, U_n \subseteq X$ *are open, then* $\bigcap_{i=1}^n U_i \subseteq X$ *is open,*

(iii) *If* $(U_i)_{i \in I}$ *is a family of sets open in* X*, then* $\bigcup_{i \in I} U_i \subseteq X$ *is open.*

Proof: The statements are easy to show. ∎

Corollary 1.7: (i) $\{\}$ *and* X *are closed,*

(ii) *If* $X_1, \ldots, X_n \subseteq X$ *are closed, then* $\bigcup_{i=1}^n X_i \subseteq X$ *is closed,*

(iii) *If* $(X_i)_{i \in I}$ *is a family of sets closed in* X*, then* $\bigcap_{i \in I} X_i \subseteq X$ *is closed.*

Proof: Follows immediately with Lemma 1.6 and De Morgans laws. ∎

Remark 1.8: Arbitrary unions of closed sets respectively arbitrary intersections of open sets need not be closed respectively open, e. g. let $(X,d) = (\mathbb{R}, d_{|\cdot|})$, then

$$\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\} \quad , \quad \bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1\right] = (0,1].$$

Definition 1.9: Let $A \subset X$. We call $\operatorname{Int}(A) := \{x \in A : \exists U \in \mathfrak{U}(x) : U \subseteq A\}$ the *interior of* A, $\operatorname{cl}(A) := \overline{A} := \{x \in X : \forall U \in \mathfrak{U}(x) : U \cap A \neq \{\}\}$ the closure of A and $\partial A = \{x \in X : \forall U \in \mathfrak{U}(x) : U \cap A \neq \{\} \neq U \cap (X \setminus A)\}$ the *boundary of* A.

Lemma 1.10: *Let $A, B \subseteq X$. Then we have:*

(i) $\mathrm{Int}(A) = \bigcup_{U\subset X \text{ open}, U\subset A} U$ *is the biggest (with respect to inclusion) open subset contained in A.*

(ii) $\mathrm{cl}(A) = \bigcap_{F\subseteq X \text{ closed}, F\supseteq A} F$ *is the smallest closed set containing A,*

(iii) $\partial A \subseteq X$ *is closed and we have* $\mathrm{cl}(A) = A \cup \partial A$, $\mathrm{Int}(A) = A \cap (X \setminus \partial A)$ *and* $\partial A = \mathrm{cl}(A) \setminus \mathrm{Int}(A)$,

(iv) *If $A \subseteq B$, then* $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$,

(v) *If $A \subseteq B$, then* $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,

(vi) $A \subseteq X$ *is open if and only if* $A = \mathrm{Int}(A)$,

(vii) $A \subseteq X$ *is closed if and only if* $A = \mathrm{cl}(A)$,

(viii) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$, $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$.

The equalities in (vi) do not hold for the respective other operation "$\cap$", "$\cup$".

Proof: (i) "$\supseteq$": Let $U \subseteq X$ be open with $U \subseteq A$, then $U \subseteq \mathrm{Int}(A)$ because $\forall x \in U : U \in \mathfrak{U}(x)$ and $U \subseteq A$. "$\subseteq$": That $\mathrm{Int}(A) \subseteq A$ is clear. $\mathrm{Int}(A) \subseteq X$ is open, if $\forall x \in \mathrm{Int}(A) : \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A$, which holds since $\forall x \in \mathrm{Int}(A)$ there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.

(ii) "$\supseteq$": Let $x \in \bigcap_{F\subseteq X \text{ closed}, F\supseteq A} F$. Let $U \in \mathfrak{U}(x)$ with $U \cap A = \{\}$. Without loss of generality let $U \subseteq X$ open. Then $X \setminus U \subseteq X$ is closed with $X \setminus U \supseteq A$ and $x \notin X \setminus U$, which is a contradiction.
"$\subseteq$": Let $x \in \mathrm{cl}(A)$ and let $F \subseteq X$ be closed with $F \supseteq A$. If $x \notin F$, then $X \setminus F \in \mathfrak{U}(x)$ with $X \setminus F \cap A = \{\}$, which is a contradiction.

(iii) "$\partial A = \mathrm{cl}(A) \setminus \mathrm{Int}(A)$": $x \in \partial A$ if and only if $\forall U \in \mathfrak{U}(x) : U \cap A \neq \{\}$ and $U \cap X \setminus A \neq \{\}$ which holds if and only if $x \in \mathrm{cl}(A)$ and $x \notin \mathrm{Int}(A)$. In particular, $\partial A \subseteq X$ is closed.
"$\mathrm{cl}(A) = A \cup \partial A$": The inclusion $A \cup \partial A \subseteq \mathrm{cl}(A)$ is obvious. Let $x \in \mathrm{cl}(A)$ and $x \notin A$, then we have: $\forall U \in \mathfrak{U}(x) : U \cap A \neq \{\}$ and $U \cap (X \setminus A) \neq \{\}$. But then, $x \in \partial A$ which was to be shown.
"$\mathrm{Int}(A) = A \cap (X \setminus \partial A)$": The inclusion $\mathrm{Int}(A) \subseteq A \cap (X \setminus \partial A)$ is clear. Let $x \in A$ and $x \notin \partial A$. Then $\exists U \in \mathfrak{U}(x) : U \cap (X \setminus A) = \{\}$, so $U \subseteq A$ and then $x \in \mathrm{Int}(A)$.

(iv) It holds that $\mathrm{Int}(A) \subseteq A \subseteq B$, so by (i) $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$.

(v) Since $A \subseteq B \subseteq \mathrm{cl}(B)$, by (ii) $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

(vi) Direct consequence of (i) and (ii).

(vii) Direct consequence of (i) and (ii).

(viii) Because $A \cup B \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$, we have $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$ by (v). Furthermore we have

$$\begin{matrix} A \subseteq \mathrm{cl}(A \cup B) \\ B \subseteq \mathrm{cl}(A \cup B) \end{matrix} \Rightarrow \begin{matrix} \mathrm{cl}(A) \subseteq \mathrm{cl}(A \cup B) \\ \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B) \end{matrix} \Rightarrow \mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B).$$

It holds that $\operatorname{Int}(A) \cap \operatorname{Int}(B) \subseteq A \cap B$, so via Lemma 1.10(iv), we know that $\operatorname{Int}(A) \cap \operatorname{Int}(B) \subseteq \operatorname{Int}(A \cap B)$. Furthermore

$$A \supseteq \operatorname{Int}(A \cap B), B \supseteq \operatorname{Int}(A \cap B) \Rightarrow \operatorname{Int}(A) \cap \operatorname{Int}(B) \supseteq \operatorname{Int}(A \cap B). \qquad \blacksquare$$

Example 1.11: (i) Let $X \neq \{\}$ be equipped with the discrete metric d. For $A \subseteq X$, we have

$$\partial A = \operatorname{cl}(A) \setminus \operatorname{Int}(A) = A \setminus A = \{\},$$

In particular, for $a \in X$ and $r = 1$: $\partial B_r(a) = \{\} \neq \{x \in X : d(x, a) = r\}$, if $\#(X) > 1$.

(ii) In $(\mathbb{R}, d_{|\cdot|})$, we have $\operatorname{cl}(\mathbb{Q}) = \mathbb{R}$, $\operatorname{Int}(\mathbb{Q}) = \{\}$, $\partial\mathbb{Q} = \mathbb{R}$ and for $a < b \in \mathbb{R}$ we have $\operatorname{cl}((a, b)) = [a, b]$ as well as $\operatorname{Int}([a, b]) = (a, b)$.

2 Convergence, Completeness and Baire's theorem

Let (X, d) be a metric space.

Definition 2.1: A sequence $(x_n)_{n\in\mathbb{N}}$ is called

(i) *convergent to* $x \in X$, if $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N} : \forall\, n \geq N : x_n \in B_\varepsilon(x)$,

(ii) *convergent*, if it converges to some $x \in X$,

(iii) a *Cauchy sequence*, if $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N} : \forall\, n, m \geq N : d(x_n, x_m) < \varepsilon$.

Lemma 2.2: *Limits in metric spaces are unique, i. e. if a sequence* $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ *converges to* $x \in X$ *and* $y \in X$, *then* $x = y$.

Proof: Because $x_n \to x$, $y_n \to y$, we have $0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0$, so $d(x, y) = 0$ and that implies $x = y$. ■

Definition 2.3: For $Y \subseteq X$, we call $d_Y := d|_{Y\times Y}$ the *relative metric* from (X, d) on Y.

Lemma 2.4: *For* $A \subseteq X$, *we have* $\mathrm{cl}(A) = \{x \in X : \exists\, (x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}} : x_n \to x\}$.

Definition 2.5: X is called *complete*, if every Cauchy sequence in X converges.

Example 2.6: (i) $(\mathbb{R}, |\cdot|)$ is complete by construction, $\mathbb{R}^n$ and $\mathbb{C}^n$ with d_p for $1 \leq p < \infty$ or d_∞ are complete.

(ii) $(\ell^\infty(M), d_M)$ is complete. Let $(f_k)_{k\in\mathbb{N}} \in (\ell^\infty)^{\mathbb{N}}$ be a Cauchy sequence. For $x \in M$, by

$$|f_k(x) - f_l(x)| \leq \|f_k - f_l\|_M = d_M(f_k, f_l)$$

for $k, l \in \mathbb{N}$, the sequence $(f_k(x))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. Because $\mathbb{C}$ is complete, $(f_k(x))_{k\in\mathbb{N}}$ converges for all $x \in M$. Let

$$\begin{aligned} f : M &\longrightarrow \mathbb{C}, \\ x &\longmapsto \lim_{k\to\infty} f_k(x). \end{aligned}$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $d_M(f_k, f_l) < \varepsilon$ for all $k, l \in\geq N$. Then we have for all $k \geq N$ and $x \in M$:

$$|f_k(x) - f(x)| = \lim_{\substack{l\to\infty \\ l\geq N}} |f_k(x) - f_l(x)| \leq \varepsilon.$$

Then, for all $k \geq N$, we have $\|f_k - f\|_M \leq \varepsilon$. In particular, for $x \in M$:

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq \varepsilon + \|f_N\|_M,$$

so $\|f\|_M \leq \varepsilon + \|f_N\|_M < \infty$, which shows that $f \in \ell^\infty(M)$ and with what we have shown above, we have $f_k \to f$ for $k \to \infty$ in d_M.

(iii) $(\mathbb{R}^{\mathbb{N}}, d)$ is complete (refer to Example 1.2 for the metric used). Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{\mathbb{N}}$ and let $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. For $k \in \mathbb{N}$, let $x^{(k)} = (x_n^{(k)})_{k\in\mathbb{N}}$.

We show that $(x^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}^{\mathbb{N}}, d)$ if and only if $(x_n^{(k)})_{k\in\mathbb{N}}$ is Cauchy in $(\mathbb{R}, d_{|\cdot|})\,\forall\, n \in \mathbb{N}$.

"$\Rightarrow$": Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\frac{1}{2^n}\frac{|x_n^{(k)} - x_n^{(k)}|}{1 + |x_n^{(k)} - x_n^{(l)}|} \leq d(x^{(k)}, x^{(l)}) < \frac{1}{2^n}\frac{\varepsilon}{1+\varepsilon}$$

for all $k, l \geq N$. Then, for all $k, l \geq N$, it holds that $|x_n^{(k)} - x_n^{(l)}| < \varepsilon$ by the increasing function used in Example 1.2.

"$\Leftarrow$": Let $\varepsilon > 0$. Choose $K \in \mathbb{N}$ such that $\sum_{n=K+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$. Choose $N \in \mathbb{N}$, such that for all $k, l \geq N$ and for all $0 \leq n \leq K$:

$$|x_n^{(k)} - x_n^{(l)}| < \frac{\varepsilon}{4}$$

Then, for all $k, l \geq N$ we have

$$\begin{aligned} d(x^{(k)}, x^{(l)}) &= \sum_{n=0}^{k} \frac{1}{2^n}\frac{|x_n^{(k)} - x_n^{(l)}|}{1 + |x_n^{(k)} - x_n^{(l)}|} + \sum_{n=K+1}^{\infty} \frac{1}{2^n} \\ &< \Big(\sum_{n=0}^{K} \frac{1}{2^n}\Big)\frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Similarily we show: $x^{(k)} \to x$ for $k \to \infty$ if and only if $x_n^{(k)} \to x_n$ for $k \to \infty$ in $(\mathbb{R}, d_{|\cdot|})\ \forall n \in \mathbb{N}$. Thus, $(\mathbb{R}^{\mathbb{N}}, d)$ is complete since $(\mathbb{R}, |\cdot|)$ is complete.

(iv) $(C^1([0,1]), d_1)$ from Examples 1.2 v) is complete by Analysis I. Inductively: $(C^n([0,1]), d_n)$ is complete.

(v) (ℓ^p, d_p) is complete for $1 \leq p < \infty$.

(vi) Let $X \neq \{\}$ and d the discrete metric. Then, the Cauchy sequences and convergent sequences are exactly the eventually constant sequences. Thus, (X, d) is complete.

(vii) $\mathbb{Q} \subseteq \mathbb{R}$ and $\mathbb{R}^{\times} \subseteq \mathbb{R}$ with the corresponding relative metrics are not complete.

Proposition 2.7: *For $x, y, x', y' \in X$ we have*

$$|d(x,y) - d(x',y')| \leq d(x,x') + d(y,y').$$

The proof is obvious, you need the triangular equality and you have to consider two cases.

Definition 2.8: A mapping $j : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called an isometry, if

$$d_Y(j(x), j(y)) = d_X(x, y)$$

for all $x, y \in X$.

Remark 2.9: For an isometry $j : X \to Y$ we will identify X with $j(X) \subseteq Y$ as metric spaces and write $X \subseteq Y$. This is justified, since isometries preserve all properties of metric spaces, e. g. open and closed sets, convergence and Cauchy sequences, bounded sets and continuity are preserved under isometries.

Theorem 2.10: *Let (X, d) be a metric space. Then there is a complete metric space $(X_\sim, d_\sim)$ such that $X \subseteq X_\sim$, $d_\sim|_{X\times X} = d$ and $X \subseteq X_\sim$ dense (that is $\mathrm{cl}(X) = X_\sim$).*

Proof: Let

$$X_\sim = \{(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \text{ is a Cauchy sequence}\}/\sim$$

with respect to the equivalence relation $\sim$ defined by

$$(x_k)_{k\in\mathbb{N}} \sim (y_k)_{k\in\mathbb{N}} :\Leftrightarrow \lim_{k\to\infty} d(x_k, y_k) = 0.$$

Furthermore, let

$$\begin{aligned} d_\sim : X_\sim \times X_\sim &\longrightarrow \mathbb{R} \\ d_\sim([(x_k)_{k\in\mathbb{N}}], [(y_k)_{k\in\mathbb{N}}]) &= \lim_{k\to\infty} d(x_k, y_k). \end{aligned}$$

In the following, we will denote $[x_k] := [(x_k)_{k\in\mathbb{N}}]$. $d_\sim$ is well-defined: First of all, by Proposition 2.7, we have

$$\forall\, k, l \in \mathbb{N} : |d(x_k, y_k) - d(x_l, y_l)| \leq d(x_k, x_l) + d(y_k, y_l),$$

so $(d(x_k, y_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}, d_{|\cdot|})$, thus convergent. Secondly, for $[x_k] = [\tilde{x}_k]$ and $[y_k] = [\tilde{y}_k]$ we have, again by Proposition 2.7, that

$$\forall\, k \in \mathbb{N} : 0 \leq |d(x_k, y_k) - d(\tilde{x}_k, \tilde{y}_k)| \leq d(x_k, \tilde{x}_k) + d(y_k, \tilde{y}_k) \overset{k\to\infty}{\longrightarrow} 0,$$

so $\lim_{k\to\infty} d(x_k, x_k) = \lim_{k\to\infty} d(\tilde{x}_k, \tilde{y}_k)$. Finally, $d_\sim$ is non-negative, symmetric and fulfills the triangular inequality since d does. $d_\sim$ is positive definite by the definition of "$\sim$".

So $(X_\sim, d_\sim)$ is a metric space. Now, let

$$\begin{aligned} j : X &\longrightarrow X_\sim, \\ x &\longmapsto [(x)_{k\in\mathbb{N}}]. \end{aligned}$$

Then j is an isometry, because for all $x, y \in X$:

$$d_\sim(j(x), j(y)) = d_\sim([(x)_{k\in\mathbb{N}}], [(y)_{k\in\mathbb{N}}]) = \lim_{k\to\infty} d(x, y) = d(x, y).$$

Hence, we can identify $X \cong j(X)$ and $d_\sim|_{X\times X} = d$.

Let $\xi = [x_k] \in X_\sim$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $d(x_k, x_l) < \frac{\varepsilon}{2}$ for all $k, l \geq N$. For all $k \in \mathbb{N}$, the sequence $(d(x_k, x_l))_{l\in\mathbb{N}}$ converges in $(\mathbb{R}, d_{|\cdot|})$ due to Proposition 2.7. In particular, for all $k \geq N$, we have that

$$\lim_{\substack{l\to\infty \\ l\geq N}} d(x_k, x_l) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus, for all $k \geq N$:

$$d_\sim(j(x_k), \xi) = \lim_{l\to\infty} d(x_k, x_l) \overset{l\to\infty}{\longrightarrow} 0,$$

so $j(x_k) \to \xi$ for $k \to \infty$ with respect to $d_\sim$ and $j(X) \subseteq X_\sim$ is dense.

Let $(\xi^{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in $(X_\sim, d_\sim)$. Then for all $k \in \mathbb{N}$ there exists $x_k \in X$ such that $d_\sim(\xi^{(k)}, j(x_k)) < \frac{1}{k+1}$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$d_\sim(\xi^{(k)}, \xi^{(l)}) < \frac{\varepsilon}{3} \quad , \quad \frac{1}{k+1} < \frac{\varepsilon}{3}$$

holds for all $k, l \geq N$. Hence, for all $k, l \geq N$

$$\begin{aligned} d(x_k, x_l) = d_\sim(j(x_k), j(x_l)) &\leq d_\sim(j(x_k), \xi^{(k)}) + d_\sim(\xi^{(k)}, \xi^{(l)}) + d_\sim(\xi^{(l)}, j(x_l)) \\ &< \frac{1}{k+1} + \frac{\varepsilon}{3} + \frac{1}{l+1} < \varepsilon, \end{aligned}$$

thus $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. Let now $\xi = [x_k]$. For all $k \in \mathbb{N}$:

$$d_\sim(\xi^{(k)}, \xi) \leq d_\sim(\xi^{(k)}, j(x_k)) + d_\sim(j(x_k), \xi) \overset{k\to\infty}{\longrightarrow} 0,$$

so $\xi^{(k)} \to \xi$ for $k \to \infty$ with repect to $d_\sim$, so $(X_\sim, d_\sim)$ is complete. ■

Lemma 2.11: *Let $Y \subseteq X$. Then we have: If (Y, d_Y) is complete, then $Y \subseteq X$ is closed. If (X, d) is complete, then equivalence holds.*

Proof: "⇒": Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Y with $x_n \to x \in X$ for $n \to \infty$ with respect to d. Then, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d) and thus a Cauchy sequence in (Y, d_Y), hence there exists a $y \in Y$, such that $x_n \to y$ for $n \to \infty$ with respect to d_Y. But then $x_n \to y$ for $n \to \infty$ with respect to d holds, and thus $x = y \in Y$. By Lemma 2.4, $Y \subseteq X$ is closed.

"⇐": Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (Y, d_Y). Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d) and because X is complete, there exists an $x \in X$, such that $x_n \to x$. By Lemma 2.4, $x \in Y$ and thus $x_n \to y$ for $n \to \infty$ with respect to d_Y. ■

Definition 2.12: For $\{\} \neq A \subseteq X$, we call

$$\operatorname{diam}(A) = \sup\{d(x,y) \mid x, y \in A\} \in [0, \infty]$$

the *diameter of* A.

Theorem 2.13 (Cantor's Intersection Theorem): *Let (X, d) be a complete metric space and $(A_n)_{n\in\mathbb{N}}$ be a sequence of non-empty, closed subsets $A_k \subseteq X$ with $A_{k+1} \subseteq A_k$ for all $k \in \mathbb{N}$ and $\operatorname{diam}(A_k) \to 0$ for $k \to \infty$. Then there is an $x \in X$, such that $\bigcap_{k\in\mathbb{N}} A_k = \{x\}$.*

Proof: For $k \in \mathbb{N}$, choose $x_k \in A_k$. Due to

$$d(x_k, x_l) \leq \operatorname{diam}(A_{\min\{k,l\}})$$

for all $n, l \in \mathbb{N}$ and $\operatorname{diam}(A_k) \to 0$ we know that $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in X, hence there is an $x \in X$ such that $x_k \to x$ for $k \to \infty$ with respect to d. For all $n \in \mathbb{N}$, we have $x = \lim_{k\to\infty, k\geq n} x_k \in \operatorname{cl}(A_n) = A_n$ due to Lemma 2.4 and the properties of $(A_k)_{k\in\mathbb{N}}$. Thus $x \in \bigcap_{k\in\mathbb{N}} A_k$.

Let now $y \in \bigcap_{k\in\mathbb{N}} A_k$, then $0 \leq d(x, y) \leq \operatorname{diam}(A_k)$ for all $k \in \mathbb{N}$, hence $d(x, y) = 0$ and $x = y$. ■

Theorem 2.14 (Baire's Theorem): *Let (X, d) be a complete metric space, $(F_n)_{n\in\mathbb{N}}$ a sequence of closed subsets $F_n \subseteq X$. Then we have: If $\operatorname{Int}(\bigcup_{n\in\mathbb{N}} F_n) \neq \{\}$, then there exists an $n_0 \in \mathbb{N} : \operatorname{Int}(F_{n_0}) \neq \{\}$.*

Proof: We will first argue for the following claim: If $F \subseteq X$ is a closed subset with $\operatorname{Int}(F) = \{\}$, then

$$\forall x \in X \, \forall r > 0 \, \exists x_1 \in X \, \exists r_1 > 0 : \overline{B}_{r_1}(x_1) \subseteq (X \setminus F) \cap B_r(x) \tag{2.1}$$

holds. This is true, because $(X \setminus F) \cap B_r(x)$ is open and non-empty due to $\operatorname{Int}(F) = \{\}$ for all $x \in X$, $r > 0$.

Now let $(F_n)_{n\in\mathbb{N}}$ be as in the assumption. We assume that $\operatorname{Int}(F_n) = \{\} \, \forall n \in \mathbb{N}$. Let $x_0 \in X$, $r_0 > 0$ such that $B_{r_0}(x_0) \subseteq \bigcup_{n\in\mathbb{N}} F_n$. Then, by (2.1), there exist $x_1 \in X$, $r_1 \in (0, 1)$, such that $\overline{B}_{r_1}(x_1 \subseteq X \setminus F_0 \cap B_{r_0}(x_0)$. Again, by (2.1) there exist $x_2 \in X$, $r_2 \in (0, \frac{1}{2})$, such that $B_{r_2}(x) \subseteq (X \setminus F_1) \cap B_{r_1}(x_1)$.

Inductively, we find a sequence $(B_n)_{n\in\mathbb{N}\setminus\{0\}}$ $(B_1 := \overline{B}_{r_1}(x_1), B_2 := \overline{B}_{r_2}(x_2), \ldots)$ of closed balls $B_n \subseteq X$ such that $B_{n+1} \subseteq B_n \subseteq X \setminus F_{n-1}$ for all $n \geq 1$ and $\operatorname{diam}(B_n) \to 0$ for $n \to \infty$. By Theorem 2.13 there exists $x \in X$ such that

$$x \in \bigcap_{n\in\mathbb{N}\setminus\{0\}} B_n \subseteq \bigcap_{n\in\mathbb{N}\setminus\{0\}} X \setminus F_{n-1} = X \setminus \bigcup_{n\in\mathbb{N}} F_n.$$

On the other hand, $x \in B_1 \subseteq B_{r_0}(x_0) \subseteq \bigcup_{n\in\mathbb{N}} F_n$, which is a contradiction. ■

Definition 2.15: Let $M \subseteq X$. Then M is called *nowhere dense*, if $\operatorname{Int}(\operatorname{cl}(M)) = \{\}$.
Countable unions of nowhere dense sets are called *meagre*.

Corollary 2.16: *Let (X, d) be a complete metric space, $M \subseteq X$ meagre. Then $\operatorname{Int}(M) = \{\}$.*

Proof: Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of nowhere dense sets such that $M = \bigcup_{n\in\mathbb{N}} M_n$. We assume, that $\operatorname{Int}(\bigcup_{n\in\mathbb{N}} M_n) \neq \{\}$. Then, $\operatorname{Int}(\bigcup_{n\in\mathbb{N}} \operatorname{cl}(M_n)) \neq \{\}$, hence by Theorem 2.14, there exists $n_0 \in \mathbb{N}$ such that $\operatorname{Int}(\operatorname{cl}(M_{n_0})) \neq \{\}$, which is a contradiction. ∎

Lemma 2.17: *Let $A \subseteq X$. Then $\operatorname{cl}(A) = X$ holds if and only if $\operatorname{Int}(X \setminus A) = \{\}$.*

Proof: "$\Rightarrow$": Assume, that there exists $x \in \operatorname{Int}(X \setminus A)$. Then, for a sequence $(x_k)_{k\in\mathbb{N}}$ in A with $x_k \to x$ for $k \to \infty$ we see that $U \cap A = \{\}$ for all $U \in \mathfrak{U}(x)$.
"$\Leftarrow$": Let $x \in X$. Then for $k \in \mathbb{N} \setminus \{0\}$, there is $x_k \in B_{\frac{1}{k}}(x) \cap A$. Then $(x_k)_{k\in\mathbb{N}}$ is a sequence in A which converges to x. ∎

Corollary 2.18: *Let (X, d) be a complete metric space and $(U_n)_{n\in\mathbb{N}}$ a sequence of open, dense subsets $U_n \subseteq X$. Then, $\bigcap_{n\in\mathbb{N}} U_n \subseteq X$ is also dense.*

Proof: For all $n \in \mathbb{N}$, we have $U_n \subseteq X$ open and $\operatorname{cl}(U_n) = X$. By Lemma 2.17, for all $n \in \mathbb{N}$, $X \setminus U_n \subseteq X$ is closed and non-empty and $\operatorname{Int}(X \setminus U_n) = \{\}$. By Theorem 2.14 $\operatorname{Int}(X \setminus \bigcap_{n\in\mathbb{N}} U_n) = \operatorname{Int}(\bigcup_{n\in\mathbb{N}} X \setminus U_n)) = \{\}$, so by Lemma 2.17, $\bigcap_{n\in\mathbb{N}} U_n \subseteq X$ is dense. ∎

Remark 2.19: In general metric spaces, Baire's Theorem need not hold. Let for example $(X, d) = (\mathbb{Q}, d_{|\cdot|,\mathbb{Q}})$ and for $q \in \mathbb{Q}$, let $\{q\} \subseteq \mathbb{Q}$ which is closed and non-empty with $\operatorname{Int}(\{q\}) = \{\}$. But $\operatorname{Int}(\bigcup_{q\in\mathbb{Q}} \{q\}) = \operatorname{Int}(\mathbb{Q}) = \mathbb{Q} \neq \{\}$.

3 Continuous mappings between metric spaces

In the following, let (X, d) and (Y, d') be metric spaces.

Definition 3.1: A mapping $f : X \to Y$ is called

(i) *continuous in* $x \in X$, if $\forall \varepsilon > 0 \, \exists \delta > 0 : f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$,

(ii) *continuous*, if it is continuous in every $x \in X$,

(iii) *sequentially continuous in* $x \in X$, if for all sequences $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ with $x_n \to x$, $f(x_n) \to f(x)$ holds,

(iv) *sequentially continuous*, if it is sequentially continuous in every $x \in X$,

(v) *uniformly continuous*, if $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X : f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Theorem 3.2: *For $f : X \to Y$, the following are equivalent:*

(i) *f is continuous,*

(ii) *f is sequentially continuous,*

(iii) *The preimage $f^{-1}(U)$ of every open set $U \subseteq Y$ is open in X,*

(iv) *The preimage $f^{-1}(A)$ of every closed set $A \subseteq Y$ is closed in X.*

Definition 3.3: A subset $K \subseteq X$ is called *compact*, if for every open cover $(U_i)_{i\in I}$ of K (i. e. $U_i \subseteq X$ open $\forall i \in I$ and $K \subseteq \bigcup_{i\in I} U_i$) there are $i_1, \ldots, i_n \in I$, such that $K \subseteq \bigcup_{j=1}^n U_{i_j}$.

Theorem 3.4: *Let $K \subseteq X$ be compact and equipped with the relative metric d_K and let $f : K \to (Y, d')$ be continuous. Then, f is uniformly continuous.*

Proof: Let $\varepsilon > 0$. For $x \in K$, choose $\delta_x > 0$ with $f(B_{\delta_x}(x)) \subseteq B_{\varepsilon/2}(f(x))$. Then, $K \subseteq \bigcup_{x\in X} B_{\delta/2}(x)$ is an open cover. Because K is compact, there are $x_1, \ldots, x_n \in K$, such that with $\delta_i := \delta_{x_i}$:

$$K \subseteq \bigcup_{i=1}^{n} B_{\delta_i/2}(x_i).$$

Let now $\delta := \min\{\delta_{i/2} \mid 1 \leq i \leq n\}$ and let $x, x' \in K$ with $d_K(x, x') < \delta$. Finally, let $i \in \{1, \ldots, n\}$, such that $x \in B_{\delta_i/2}(x_i)$. Via the triangular inequality, $x' \in B_{\delta_i}(x_i)$, thus

$$d'(f(x), f(y)) \leq d'(f(x), f(x_i)) + d'(f(x_i), f(x')) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and f is uniformly continuous. ∎

Theorem 3.5: *Let $X_0 \subseteq X$ be dense, Y complete and $f : X_0 \to Y$ uniformly continuous. Then, there is exactly one continous extension $F : X \to (Y, d')$ of f (i. e. $F|_{X_0} = f$). This extension is uniformly continuous.*

Proof: For every $\varepsilon > 0$, choose $\delta_\varepsilon > 0$ such that "$\forall\, x, y \in X_0 : d(x, y) < \delta_\varepsilon \Rightarrow d'(f(x), f(y)) < \varepsilon$" holds. Define

$$\begin{aligned} F : X &\longrightarrow Y \\ x &\longmapsto \lim_{n\to\infty} f(x_n) \end{aligned}$$

for a sequence $(x_n)_{n\in\mathbb{N}} \in X_0^{\mathbb{N}}$ with $x_n \to x$ for $n \to \infty$. X_0 is dense in X, hence there is such a sequence for all $x \in X$. For the well-definedness of F, we have to check that $F(x)$ does not depend on the sequence that converges to x and that $(f(x_n))_{n\in\mathbb{N}}$ converges for any such sequence.

Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_0, such that $x_n \to x$ for $n \to \infty$. For $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $d(x_n, x_m) < \delta_\varepsilon$ for all $n, m \geq N$. Then $d'(f(x_n), f(x_m)) < \varepsilon$ for all $n, m \geq N$. Because Y is complete, $(f(x_n))_{n\in\mathbb{N}}$ converges in (Y, d'). Now, let $(x'_n)_{n\in\mathbb{N}}$ be another sequence in X_0 with $x'_n \to x$ for $n \to \infty$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$, such that

$$d(x_N, x), d(x'_N, x) < \frac{1}{2}\delta_{\varepsilon/3} \quad , \quad d'(\lim_{n\to\infty} f(x_N), f(x_N)), d'(\lim_{n\to\infty} f(x'_N), x'_N) < \frac{\varepsilon}{3}.$$

We conclude $d(x_N, x'_N) \leq d(x_N, x) + d(x, x'_N) < \delta_{\frac{\varepsilon}{3}}$ and thus

$$\begin{aligned} &d'(\lim_{n\to\infty} f(x_n), \lim_{n\to\infty} f(x'_n)) \\ &\quad \leq d'(\lim_{n\to\infty} f(x_n), f(x_N)) + d'(f(x_N), f(x'_N)) + d(f(x'_N), \lim_{n\to\infty} f(x'_n)) \\ &\quad < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

hence $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x'_n)$.

We have $F|_{X_0} = f$ since we can choose the constant sequence $(x)_{n\in\mathbb{N}}$ in X_0 for $x \in X_0$, which converges to x.

Last, we show that F is uniformly continuous. Let $\varepsilon > 0$ and $x, y \in X$, such that $d(x, y) < \delta_{\varepsilon/2}$. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in X_0^{\mathbb{N}}$, such that $x_n \to x$ and $y_n \to y$ for $n \to \infty$. Then, $\lim_{n\to\infty} d(x_n, y_n) = d(x, y) < \delta_{\varepsilon/2}$ holds. Choose $N \in \mathbb{N}$, such that $d(x_n, y_n) < \delta_{\frac{\varepsilon}{2}}$ for all $n \geq N$. We conclude for all $n \geq N$:

$$d'(F(x), F(y)) = d'(\lim_{n\to\infty} f(x_n), \lim_{n\to\infty} f(y_n)) = \lim_{n\to\infty} d'(f(x_n), f(y_n)) \leq \frac{\varepsilon}{2} < \varepsilon,$$

thus F is uniformly continuous.

F is a unique as a continuous extension of f since every continuous extension is sequentially continuous. ■

4 Topologic spaces

Definition 4.1: Let $\{\} \neq X$ be a set.

(i) A subset $\mathfrak{T} \subseteq \mathfrak{P}(X)$ is called a *topology* on X, if

  (1) $\{\}, X \in \mathfrak{T}$,

  (2) $U_1, \dots, U_n \in \mathfrak{T} \Rightarrow \bigcap_{i=1}^n U_i \in \mathfrak{T}$,

  (3) $(U_i)_{i \in I} \in \mathfrak{T}^I \Rightarrow \bigcup_{i \in I} U_i \in \mathfrak{T}$

hold. The elements of $\mathfrak{T}$ are called *open*. $A \subseteq X$ is called *closed*, if $X \setminus A \in \mathfrak{T}$. $(X, \mathfrak{T})$ is then called a *topological space*.

(ii) A topological space $(X, \mathfrak{T})$ is called *Hausdorff* (or *separated*), if

$$\forall\, x, y \in X, x \neq y : \exists\, U, V \in \mathfrak{T} : x \in U, y \in V, U \cap V = \{\}$$

holds.

(iii) Let $\mathfrak{T}_1, \mathfrak{T}_2$ be topologies on X. $\mathfrak{T}_1$ is called *coarser* than $\mathfrak{T}_2$, if $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. $\mathfrak{T}_2$ is then called *finer* than $\mathfrak{T}_1$.

Example 4.2: (i) Let (X, d) be a metric space. By (Lemma 1.6),

$$\mathfrak{T} := \{U \subseteq X \mid U \text{ is open with respect to } d\}$$

defines a topology on X. The corresponding topological space $(X, \mathfrak{T})$ even is Hausdorff: For $x \neq y \in X$ with $x \neq y$, set $r := 2^{-1} d(x, y) > 0$. Then $B_r(x), B_r(y) \subseteq X$ are open and disjoint with $x \in B_r(x), y \in B_r(y)$.

(ii) Let $\{\} \neq X$ be a set. Then $\mathfrak{T} = \{\{\}, X\}$ is a topology on X which is not Hausdorff for $\#(X) > 1$. $\mathfrak{T}$ is called the *indiscrete topology*.

Remark 4.3: Metrics d, d' on a set X are called equivalent, if there is $C > 0$, such that

$$\frac{1}{C} d(x, y) \leq d(x, y) \leq C d'(x, y)$$

for all $x, y \in X$. In this case, we have $B_{\varepsilon/C}(y) \subseteq B'_\varepsilon(y)$ and $B'_{\varepsilon/C}(y) \subseteq B_\varepsilon(y)$, where $B'_\varepsilon(x)$ denotes the ball arround x of radius ε with respect to d'.

Definition 4.4: Let $(X, \mathfrak{T})$, $(Y, \mathfrak{T}')$ be topoligical spaces, $A \subseteq X$, $x \in X$, $f : X \to Y$ a function and $(x_n)_{n \in \mathbb{N}}$ a sequence in X.

(i) $U \subseteq X$ is called a *neighbourhood* of x, if there is $V \in \mathfrak{T}$, such that $x \in V \subseteq U$. We write

$$\mathfrak{U}(x) := \{U \subseteq X \mid U \text{ is neighbourhood of } x\}.$$

Put

$$\mathrm{cl}(A) := \{x \in X \mid \forall\, U \in \mathfrak{U}(x) : U \cap A \neq \{\}\},$$

$$\mathrm{Int}(A) := \{x \in A \mid \exists\, U \in \mathfrak{U}(x) : U \subseteq A\},$$
$$\partial A := \{x \in X \mid \forall\, U \in \mathfrak{U}(x) : U \cap A \neq \{\} \neq U \cap (X \setminus A)\},$$

cl(A) is called the *closure of* A, Int(A) is called the *interior of* A, ∂A is called the *boundary of* A.

(ii) $K \subseteq X$ is called *compact*, if for every open cover $(U_i)_{i\in I}$ of K, there are $i_1, \ldots, i_n \in I$, such that $K \subseteq \bigcup_{j=1}^{n} U_{i_j}$.

(iii) $(x_n)_{n\in\mathbb{N}}$ *converges to* x, if: $\forall\, U \in \mathfrak{U}(x) : \exists\, N \in \mathbb{N} : \forall\, n \geq N : x_n \in U$.

(iv) f is called *continuous in* x, if $\forall\, V \in \mathfrak{U}(f(x))\, \exists\, U \in \mathfrak{U}(x) : f(U) \subseteq V$. f is called *continuous* if: $\forall\, V \in \mathfrak{T}' : f^{-1}(V) \in \mathfrak{T}$. This holds if and only if f is continuous in every $x \in X$. f is called *sequentially continuous*, if $\forall\, x \in X\, \forall\, (x_n) \in X^{\mathbb{N}} : x_n \to x : f(x_n) \to f(x)$ holds.

Lemma 4.5: *Let $(X, \mathfrak{T})$ be a topological space, $A, B \subseteq X$. Then all the assertions from (1.10) hold.*

Remark 4.6: (i) Continuous functions between topological spaces are sequentially continuous, but the reverse implication doesn't hold in general.

(ii) For a topological space $(X, \mathfrak{T})$ and $Y \subseteq X$,

$$\mathfrak{T}|_Y := \{U \cap Y \mid U \in \mathfrak{T}\}$$

defines a topology on Y, the relative topology.

(iii) Compositions of continuous mappings are continuous due to

$$f^{-1}(g^{-1}(V)) = (g \circ g)^{-1}(V)$$

for $f : X \to Y$, $g : Y \to Z$ and $V \subseteq Z$.

(iv) If $\mathfrak{T}_1, \mathfrak{T}_2$ are topologies on a set $X \neq \{\}$, the identity mapping

$$\mathrm{id} : (X, \mathfrak{T}_1) \longrightarrow (X, \mathfrak{T}_2)$$

is continuous if and only if $\mathfrak{T}_2 \subseteq \mathfrak{T}_1$. In particula, there continuous bijections with non-continuous inverse functions.

Definition 4.7: A bijective mapping $f : X \to Y$ between topological spaces $(X, \mathfrak{T})$, $(Y, \mathfrak{T}')$ is called a *homeomorphism*, if f and f^{-1} are continuous.

Lemma 4.8: *Let $\{\} \neq Y \subseteq X$, $A \subseteq Y$, $K \subseteq X$. We have*

(i) *A is closed in $(Y, \mathfrak{T}|_Y)$ if and only if there is $B \subseteq X$ closed: $B \cap Y = A$,*

(ii) *A is compact in $(Y, \mathfrak{T}|_Y)$ if and only if A is compact in $(X, \mathfrak{T})$,*

(iii) *K is compact if and only if for every family $(F_i)_{i\in I}$ of sets closed in $(K, \mathfrak{T}|_K)$ with f with finite intersection property, we have $\bigcap_{i\in I} F_i \neq \{\}$.*

Proof: (i) Let $A \in \mathfrak{T}_Y$. A is closed in $(Y, \mathfrak{T}_Y)$ if and only if $\exists U \subseteq X$ open, such that $Y \setminus A = U \cap Y$ and this holds if and only if $\exists U \subseteq X$ open, such that $(X \setminus U) \cap Y = Y \setminus (U \cap Y) = A$.

(ii) "$\Rightarrow$": Let $(U_i)_{i \in I}$ be an open cover in $(X, \mathfrak{T})$. Then $A \subseteq \bigcup_{i \in I}(U_i \cap Y)$ is an open cover in $(Y, \mathfrak{T}_Y)$. Because A is compact in $(Y, \mathfrak{T}_Y)$, there exist $i_1, \dots, i_n$ such that

$$A \subseteq \bigcup_{j=1}^{n}(U_{i_j} \cap Y) \subseteq \bigcup_{j=1}^{n} U_{i_j}.$$

"$\Leftarrow$" is shown similarly to the other direction.

(iii) For a family $(U_i)_{i \in I}$ in $\mathfrak{P}(K)$ and the family of its complements $(F_i)_{i \in I} = (K \setminus U_i)_{i \in I}$ we have a collection of facts: $U_i \subseteq K$ are open for all $i \in I$ if and only if $F_i \subseteq K$ are closed for all $i \in I$. Furthermore $K = \bigcup_{i \in I} U_i$ holds if and only if $\bigcap_{i \in I} F_i = \{\}$, because if K is compact, there are $i_1, \dots, i_n \in I$, such that $K = \bigcup_{j=1}^{n} U_{i_j}$ which holds if and only if $\bigcap_{j=1}^{n} F_{i_j} = \{\}$. With these facts, (iii) follows from contraposition. ■

Lemma 4.9: *Let $\{\} \neq K \subseteq X$ be compact, $A \subseteq K$.*

(i) *If $A \subseteq K$ is closed, A is compact.*

(ii) *If $(X, \mathfrak{T})$ is Hausdorff, then $K \subseteq X$ is closed.*

(iii) *If $f : X \to Y$ is a continuous space to another topological space $(Y, \mathfrak{T}')$, then $f(K)$ is compact.*

Proof: (i) Let $(U_i)_{i \in I}$ be an open cover of A in $(K, \mathfrak{T}|_K)$. Because $A \subseteq K$ is closed in $(K, \mathfrak{T}_K)$, $K \setminus A$ is open in $(K, \mathfrak{T}|_K)$. Thus $K = \bigcup_{i \in I} U_i \cup (K \setminus A)$ is an open cover. Due to the compactness of K, there are $i_1, \dots, i_n \in I$, such that

$$K = \bigcup_{j=1}^{n} U_{i_j} \cup (K \setminus A).$$

Then $A \subseteq \bigcup_{j=1}^{n} U_{i_j}$, hence A is compact.

(ii) We will show, that $X \setminus K \subseteq X$ is open. Let $x \in X \setminus K$. For every $y \in K$, there are open neighbourhoods $U_y \in \mathfrak{U}(x)$, $V_y \in \mathfrak{U}(y)$ such that $U_y \cap V_y = \{\}$. Obviously, $K \subseteq \bigcup_{y \in K} V_y$. Since K is compact, there finitely many points $y_1, \dots, y_n \in K$, such that

$$K \subseteq \bigcup_{j=1}^{n} V_{y_j}.$$

Let $U_x := \bigcap_{j=1}^{n} U_{y_j} \in \mathfrak{U}(x)$ open in $(X, \mathfrak{T})$ with

$$U_x \cap K \subseteq \bigcap_{j=1}^{n} U_{y_j} \cap \bigcup_{k=1}^{n} V_{y_k} = \bigcup_{k=1}^{n} \left(\left(\bigcap_{j=1}^{n} U_{y_j} \right) \cap V_{y_k} \right) = \{\},$$

thus $U_x \subseteq X \setminus K$. Finally, it holds: $X \setminus K = \bigcup_{x \in X \setminus K} U_x \subseteq X$ is open.

(iii) Let $(V_i)_{i\in I}$ be an open cover of $f(K)$ in $(Y, \mathfrak{T}')$. Then $K \subseteq \bigcup_{i\in I} f^{-1}(V_i)$ is an open cover of $(X, \mathfrak{T})$. Because K is compact, there are $i_1, \dots, i_n \in I$: $K \subseteq \bigcup_{j=1}^{n} f^{-1}(V_{i_j})$, therefore

$$f(K) \subseteq \bigcup_{j=1}^{n} f(f^{-1}(V_{i_j})) \subseteq \bigcup_{j=1}^{n} V_{i_j}.$$

■

Corollary 4.10: *If $f : X \to Y$ is a bijective continuous function to a Hausdorff space $(Y, \mathfrak{T}')$ and $(X, \mathfrak{T})$ is compact, then f is a homeomorphism.*

Proof: Let $A \subseteq X$ be closed. By Lemma 4.9 (i), A then is compact. Due to Lemma 4.9 (iii), $f(A)$ is open. Now, Lemma 4.9 (ii) that $(f^{-1})^{-1}(A) = f(A) \subseteq Y$ is closed, hence $f^{-1} : Y \to X$ is continuous. ■

Remark 4.11: (i) If $\{\} \neq X$ and $\mathfrak{T}_1, \mathfrak{T}_2$ are two topologies on X such that $(X, \mathfrak{T}_1)$ is compact and $(X, \mathfrak{T}_2)$ is Hausdorff and $\mathfrak{T}_2 \subseteq \mathfrak{T}_1$, then $\mathfrak{T}_1 = \mathfrak{T}_2$.

(ii) The finer a topology, the more open and closed sets it has, the fewer compact sets, the fewer convergent sequences, the smaller its closures, the bigger its interiors, more continuous functions on the space, the less continous functions to this space.

Example 4.12: Let $X = \mathbb{R}$ (or any uncountable set) and let

$$\mathfrak{T} = \{U \subseteq X \mid U = \{\} \text{ or } X \setminus U \text{ is at most countable}\}.$$

Let $(x_n) \in X^{\mathbb{N}}$, $x \in X$. Then, $x_n \to x$ for $n \to \infty$ if and only if (x_n) is eventually constant (else, $U = X \setminus \{x_n, x_n \neq x\} \in \mathfrak{U}(x)$ is open with: $\forall N \in \mathbb{N}\, \exists n \geq N : x_n \neq x$, i. e. $x \notin U$, so (x_n) cant converge to x). In particular,

$$A = \{x \mid \exists (x_n) \in A^{\mathbb{N}} : x_n \to x\}$$

for all $A \subseteq X$, but $A \neq \mathrm{cl}(A)$ in general (e. g. for $A = \mathbb{R} \setminus \mathbb{N}$, then $X \setminus (X \setminus A) = A = \mathbb{R} \setminus \mathbb{N}$ is not at most countable, so $X \setminus A$ is not open, so $A \subseteq X$ is not closed.

5 Bases, subbases and countability

Definition 5.1: Let $(X, \mathfrak{T})$ be a topological space.

(i) A collection of sets $\mathfrak{B} \subseteq \mathfrak{T}$ is called a base of $\mathfrak{T}$, if for every $U \in \mathfrak{T}$, we have

$$U = \bigcup_{B \subseteq U, B \in \mathfrak{B}} B.$$

following the convention $\bigcup_{S \in \{\}} S = \{\}$.

(ii) A collection of sets $\mathfrak{S} \subseteq \mathfrak{T}$ is called a subbase of $\mathfrak{T}$, if

$$\mathfrak{B} = \{S_1 \cap \cdots \cap S_n \mid n \in \mathbb{N}, S_1, \ldots, S_n \in \mathfrak{S}\}$$

is a base of $\mathfrak{T}$, following the convention $\bigcap_{j=1}^{0} S_j = X$.

Example 5.2: (i) The collection of sets $\mathfrak{S} = \{(-\infty, b) \mid b \in \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}$ are a subbase for $\mathfrak{T}_{|\cdot|}$ on $\mathbb{R}$. Indeed, every open set $U \subseteq \mathbb{R}$ can be written as a union of open intervals $(a, b) \subseteq \mathbb{R}$ with $a, b \in \mathbb{R}, a < b$, which can be written as a finite intersection of elements from $\mathfrak{S}$.

(ii) The collection of sets $\mathfrak{B} = \{\prod_{i=1}^{n}(a_i, b_i) \mid a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Q}\}$ is a (countable) base for $(\mathbb{R}^n, \mathfrak{T}_{\|\cdot\|_\infty})$.

Theorem 5.3: *Let $\{\} \neq X$, $\mathfrak{S} \subseteq \mathfrak{P}(X)$. Then, there is a unique topology on X, which has $\mathfrak{S}$ as a subbase. $\mathfrak{T}$ is the coarsest topology on X which contains $\mathfrak{S}$:*

$$\mathfrak{T} = \bigcap_{\substack{\mathfrak{T}' \text{ topology on } X \\ \mathfrak{S} \subseteq \mathfrak{T}'}} \mathfrak{T}'.$$

Then $\mathfrak{T}$ is called the topology generated by $\mathfrak{S}$*.*

Proof: Let $\mathfrak{B} = \{S_1 \cap \cdots \cap S_n \mid n \in \mathbb{N}, S_1, \ldots, S_n \in \mathfrak{S}\}$ and

$$\mathfrak{T} = \left\{U \subseteq X : U = \bigcup_{B \in \mathfrak{B}, B \subseteq E} B\right\} = \left\{\bigcup_{i \in I} B_i : (B_i)_{i \in I} \in \mathfrak{B}^I, B_i \subseteq U \, \forall i \in I\right\}$$
$$= \{U \subseteq X \mid \forall x \in U : \exists B \in \mathfrak{B} : x \in B \subseteq U\}.$$

Then, $\{\}, X \in \mathfrak{T}$ by convention. That $\mathfrak{T}$ is stable under unions is clear by the second way we wrote $\mathfrak{T}$. If $U_1 = \bigcup_{i \in I} B_i^{(1)}, U_2 = \bigcup_{j \in J} B_j^{(2)} \in \mathfrak{T}$, thus

$$U_1 \cap U_2 = \bigcup_{(i,j) \in I \times J} (B_i^{(1)} \cap B_j^{(2)}) \in \mathfrak{T}.$$

By induction, $\mathfrak{T}$ is stable under finite intersections. Let $\mathfrak{T}'$ be another topology on X with $\mathfrak{S} \subseteq \mathfrak{T}'$. Then, due to the properties of a topology, we have $\mathfrak{B} \subseteq \mathfrak{T}'$ and therefore $\mathfrak{T} \subseteq \mathfrak{T}'$. If $\mathfrak{S}$ is even a subbase of $\mathfrak{T}'$, similarly $\mathfrak{T}' \subseteq \mathfrak{T}$ holds. ■

Definition 5.4: Let $(X, \mathfrak{T})$ be a topological space.

(i) Let $x \in X$. Then $\mathfrak{B} \subseteq \mathfrak{U}(x)$ is called a *neighbourhood base at* x, if: $\forall U \in \mathfrak{U}(x) \, \exists B \in \mathfrak{B} : x \in B \subseteq U$.

(ii) $(X, \mathfrak{T})$ is called *first-countable*, if every point has a countable neighbourhood base.

(iii) $(X, \mathfrak{T})$ is called *secound-countable*, if it has a countable base.

(iv) $(X, \mathfrak{T})$ is called *separable*, if there is a countable set $M \subseteq X$ with $\mathrm{cl}(M) = X$.

Lemma 5.5: *Let $(X, \mathfrak{T})$ be a topological space, $\mathfrak{B}$ a base of $\mathfrak{T}$ and $x \in X$. Then*

$$\mathfrak{B}(x) := \{B \in \mathfrak{B} \mid x \in B\}$$

is a neighbourhood base at x.

Proof: Let $U \in \mathfrak{U}(x)$. Then there exists $V \subseteq X$ open, such that $x \in V \subseteq U$. Because $\mathfrak{B}$ is a base of $\mathfrak{T}$, there exists $B \in \mathfrak{B}$, such that $x \in B \subseteq V \subseteq U$. So, $B \in \mathfrak{B}(x)$ with $x \in B \subseteq U$. ■

Lemma 5.6: (i) *Second-countable topological spaces are first-countable,*

(ii) *Metric spaces are first-countable,*

(iii) *A metric space is second-countable if and only if X is seperable (for a topological space, "$\Leftarrow$" doesn't hold!).*

(iv) *If $(X, \mathfrak{T})$ is a second-countable topological space, $(U_i)_{i \in I}$ a family in $\mathfrak{T}$. Then, there is a sequence $(U_n)_{n \in \mathbb{N}}$ in $\mathfrak{T}$, such that*

$$\bigcup_{n \in \mathbb{N}} U_n = \bigcup_{i \in I} U_i.$$

(Lindelöf's Theorem)

(v) *If $(X, \mathfrak{T})$ is second-countable and $\mathfrak{B}$ is a base of $\mathfrak{T}$, there exists $\mathfrak{B}' \subseteq \mathfrak{T}$, which is a countable base for $\mathfrak{T}$ with $\mathfrak{B}' \subseteq \mathfrak{B}$.*

Proof: (i) See Lemma 5.5.

(ii) For $x \in X$, the collection of sets $\{B_{\frac{1}{k}}(x) \mid k \in \mathbb{N}\}$ is a neighbourhood base at x.

(iii) "$\Rightarrow$": Let $\mathfrak{B} \in \mathfrak{T}$ be a countable base. For $\{\} \neq B \in \mathfrak{B}$, we choose $x_B \in B$ and put $M = \{x_B \mid \{\} \neq B \in \mathfrak{B}\}$. M then is countable. Let $x \in X$ and $U \in \mathfrak{U}(x)$, then there is $B \in \mathfrak{B} : x \in B \subseteq U$, hence $x_B \in B \cap M \subseteq U \cap M$, and thus $\mathrm{cl}(M)^{\mathfrak{T}} = X$.
"$\Leftarrow$": Now, let (X, d) be a metric space and $M \subseteq X$ countable and dense. Because M is countable, we can write $M = \{x_n \mid n \in \mathbb{N}\}$. Then

$$\mathfrak{B} := \{B_{\frac{1}{k}}(x_n) \mid n \in \mathbb{N}, k \in \mathbb{N} \setminus \{0\}\}$$

is countable. Let $\{\} \neq U \subseteq X$ open and $x \in U$. Then there is $k \in \mathbb{N} \setminus \{0\}$, such that $B_{\frac{1}{k}}(x) \subseteq U$, thus $\exists\, n \in \mathbb{N} : x_n \in B_{\frac{1}{2k}}(x)$. Then $x \in B_{\frac{1}{2k}}(x_n) \subseteq U$ (the inclusion holds due to the triangular inequality), thus $\mathfrak{B}$ is a base for $\mathfrak{T}$.

(iv) Let $(X, \mathfrak{T})$ be second-countable and $\mathfrak{B} = \{B_k \mid k \in \mathbb{N}\}$ a countable base for $\mathfrak{T}$. Let $\{\} \neq \mathfrak{U} \subseteq \mathfrak{T}$. Set

$$J = \{k \in \mathbb{N}\, \exists\, U \in \mathfrak{U} : B_k \subseteq U\} \subseteq \mathbb{N}.$$

For $k \in J$, choose $U_k \in \mathfrak{U}$ with $B_k \subseteq U_k$. We will show, that $\bigcup_{k\in\mathbb{N}} U_n = \bigcup_{U\in\mathfrak{U}} U$.

"$\subseteq$": This inclusion is clear. "$\supseteq$": Let $U \in \mathfrak{U}$ and $x \in U$. Then there is $k \in \mathbb{N}$, such that $x \in B_k \subseteq U$, hence there is a $k \in J$, such that $x \in B_k \subseteq U_k \subseteq \bigcup_{n\in\mathbb{N}} U_k$.

(v) Let $(X, \mathfrak{T})$ a topological space, $\mathfrak{B}$ a base for $\mathfrak{T}$ and $\mathfrak{B}' = \{B_k \mid k \in \mathbb{N}\}$ a countable base for $\mathfrak{T}$. For all $k \in \mathbb{N}$, we have that

$$B_k = \bigcup_{\substack{B\in\mathfrak{B} \\ B\subseteq B_k}} B.$$

Via (iv), for all $k \in \mathbb{N}\, \exists\, (B_k^{(n)})_{n\in\mathbb{N}}$ in B such that $B_k = \bigcup_{n\in\mathbb{N}} B_k^{(n)}$. Now, for all $U \in \mathfrak{T}$, we can write U as

$$U = \bigcup_{\substack{k\in\mathbb{N} \\ B_k\subseteq U}} B_k = \bigcup_{\substack{k\in\mathbb{N} \\ B_k\subseteq U}} \bigcup_{n\in\mathbb{N}} B_k^{(n)},$$

hence $\mathfrak{B}_0 = \{B_k^{(n)} \mid n, k \in \mathbb{N}\} \subseteq \mathfrak{B}$ is a countable base for $\mathfrak{T}$. ■

Corollary 5.7: *Let (X, d) be a separable metric space, $M \subseteq X$ any subset. Then $(M, d|_M)$ is separable.*

Proof: If (X, d) is separable, (X, d) is secound-countable via Lemma 5.6(iii). If $\mathfrak{B}$ is a base for (X, d), $\mathfrak{B} \cap M = \{B \cap M \mid B \in \mathfrak{B}\}$ is a base for $(M, d|_M)$. Then $(M, d|_M)$ is second-countable and via Lemma 5.6 (iii), $(M, d|_M)$ is separable. ■

Lemma 5.8: *Let $(X, \mathfrak{T})$ be first-countable, $(Y, \mathfrak{T}')$ another topological space, $A \subseteq X$. Let $f : X \to Y$ be a mapping. Then the following holds:*

(i) $\operatorname{cl}(A) = \{x \in X \mid \exists\, (x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}} : x_n \to x\}$,

(ii) *f is sequentially continuous if and only if f is continuous.*

Proof: (i) "$\supseteq$": Let $x = \lim_{n\to\infty} x_n$ for a sequence $(x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$, without loss of generality $U \in \mathfrak{U}(x)$ open. Then there is $n \in \mathbb{N} : x_n \in U \cap A$, hence $x \in \operatorname{cl}(A)$.

"$\subseteq$": Let $x \in \operatorname{cl}(A)$ and let $\mathfrak{B}(x) = \{B_k(x) \mid k \in \mathbb{N}\}$ be a countable neighbourhood base at x. Then there is $k \in \mathbb{N}$, such that $\exists\, x_k \in B_0(x) \cap \cdots \cap B_k(x) \cap A$. Then, let $U \in \mathfrak{U}(x)$ and $N \in \mathbb{N}$ such that $B_N(x) \subseteq U$, thus $\forall\, k \geq N : x_k \in B_0(x) \cap \cdots \cap B_k(x) \subseteq B_N(x) \subseteq U$, and thus $x_k \to x$ for $k \to \infty$.

(ii) "$\Leftarrow$": Refer to (Remark 4.6) (i) and (Theorem 3.2).

"$\Rightarrow$": Let $x \in X$ and $V \in \mathfrak{U}(f(x))$. Let's assume $\forall U \in \mathfrak{U}(x) : f(U) \not\subset V$. Let $\mathfrak{B}(x) = \{B_k(x) \mid k \in \mathbb{N}\}$ be a countable neighbourhood base at x. Then $\forall k \in \mathbb{N} : \exists x_k \in B_0(x) \cap \cdots \cap B_k(x)$, such that $f(x_k) \notin V$ held. As in (i), this sequence $(x_k)_{k\in\mathbb{N}}$ converged to x, but $f(x_k) \notin V$ for all $k \in \mathbb{N}$, thus $f(x_k) \not\to f(x)$ for $k \to \infty$ held, which is a contradiction. ∎

Example 5.9: (i) $\mathbb{R}^n$ and $\mathbb{C}^n$ with $\mathfrak{T}_{\|\cdot\|}$ are separable due to $\mathbb{R}^n = \mathrm{cl}(\mathbb{Q}^n)$ and $\mathbb{C} = \mathrm{cl}(\mathbb{Q} + \mathrm{i}\mathbb{Q})$.

(ii) $\mathbb{R}^{\mathbb{N}}$ with the metric as in (Example 1.2) (iv) is separable, because

$$M = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}} \mid \exists\, N \in \mathbb{N} : x_n = 0 \,\forall\, n \geq N\}$$

is countable (via $\bigcup_{n\in\mathbb{N}} \mathbb{Q}^{\mathbb{N}}$) and dense in $(\mathbb{R}^{\mathbb{N}}, d)$: Let $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and for $n \geq 1$ and $k = 0, \ldots, n$, choose

$$x_k^{(n)} \in \mathbb{Q} \cap \left(x_k - \frac{1}{n}, x_k + \frac{1}{n}\right)$$

and for $n \geq 1$ let

$$x^{(n)} = (x_0^{(n)}, \ldots, x_n^{(n)}, 0, \ldots, 0) \in M$$

with $x^{(n)} \to x$ with respect to d where $n \to \infty$ by (Example 2.6), thus $\mathrm{cl}(M) = \mathbb{R}^{\mathbb{N}}$.

(iii) For any $1 \leq p < \infty$, (ℓ^p, d_p) is separable with

$$M = \{(x_n)_{n\in\mathbb{N}} \in (\mathbb{Q} + \mathrm{i}\mathbb{Q})^{\mathbb{N}} \mid \exists\, N \in \mathbb{N} : x_n = 0 \,\forall\, n \geq N\} \subseteq \ell^p$$

is dense and countable.

(iv) Let $\{\} \neq X$ be equipped with the discrete metric. Then $\mathrm{cl}(A) = A$ for all $A \subseteq X$, thus X is countable if and only if X is countable.

(v) Let $\{\} \neq M$ be a set. Then $(\ell^\infty(M), d_M)$ is separable if and only if M is finite.

Proof: "$\Rightarrow$": Let $M = \{m_1, \ldots, m_n\}$ and

$$\begin{aligned} \Phi : (\ell^\infty(M), d_M) &\longrightarrow (\mathbb{C}^n, d_\infty) \\ f &\longmapsto (f(m_i))_{1\leq i\leq n} \end{aligned}$$

is an isometric isomorphism. Thus $(\ell^\infty(M), d_M)$ is separable by (i).

"$\Leftarrow$": Let M be infinite, $(x_n)_{n\in\mathbb{N}}$ in M with $x_n \neq x_m$ for $n \neq m$. Set

$$L := \{f \in \ell^\infty(M) \mid f(M) \subseteq \{0,1\}, f|_{M\setminus\{x_n|n\in\mathbb{N}} = 0\}.$$

Since

$$\mathfrak{P}(N) \hookrightarrow L$$

$$A \mapsto \left(f : M \to \mathbb{C}, x \mapsto \begin{cases} 1 & \exists n \in A \subseteq \mathbb{N} : x = x_n, \\ 0 & \text{else.} \end{cases} \right)$$

Then L is uncountable. Now, $d_M|_L$ is the discrete metric, hence $(L, d_M|_L)$ is not separable and therefore $\ell^\infty(M)$ is not separable. ■

6 Product topologies

Theorem 6.1: *Let $\{\} \neq X$ be a set, $(X_i, \mathfrak{T}_i)_{i\in I}$ a family of topological spaces. For $i \in I$, let $f_i : X \to X_i$ be a function. Then, there is a topolgoy $\mathfrak{T}$ on X with subbase*

$$\mathfrak{S} = \{f_i^{-1}(U) \mid U \in \mathfrak{T}_i, i \in I\}$$

Then the following statements hold:

(i) *$\mathfrak{T}$ is the coarsest topology, for which all the $f_i : (X, \mathfrak{T}) \to (X_i, \mathfrak{T}_i)$ are continuous.*

(ii) *If $(Y, \mathfrak{T}')$ is another topological space, then a function $g : (Y, \mathfrak{T}') \to (X, \mathfrak{T})$ is continuous if and only if $f_i \circ g : (Y, \mathfrak{T}') \to (X_i, \mathfrak{T}_i)$ is continuous for all $i \in I$.*

Proof: (i) That the f_i are continuous for all $i \in I$ is clear. Let $\mathfrak{U}$ be another topology on X with this property, then $\mathfrak{S} \subseteq \mathfrak{U}$ holds and thus $\mathfrak{T} \subseteq \mathfrak{U}$.

(ii) "$\Rightarrow$": This is clear via (Remark 4.6)(iii).
"$\Leftarrow$": Check that $\mathfrak{U} := \{U \subseteq X \mid g^{-1}(U) \in \mathfrak{T}'\}$ defines a topology on X. Because all compositions $f_i \circ g$ are continuous, $\mathfrak{S} \subseteq \mathfrak{U}$ holds ($g^{-1}(f_i^{-1}(U)) = (f_i \circ g)^{-1}(U)$ holds). Since $g : (Y, \mathfrak{T}') \to (X, \mathfrak{U})$ is continuous, $g : (Y, \mathfrak{T}') \to (X, \mathfrak{T})$ is continuos, too. ■

Definition 6.2: In the context of Theorem 6.1, $\mathfrak{T}$ is called the *weak topolgy* generated by the f_i for $i \in I$.

Definition 6.3: Let $(X_i, \mathfrak{T}_i)_{i\in I}$ be a family of topological spaces, $X = \prod_{i\in I} X_i$. Then, the weak topolgy generated by the projections ($j \in I$)

$$\begin{aligned} \pi_j : X &\longrightarrow X_j \\ (x_i)_{i\in I} &\longmapsto x_j \end{aligned}$$

is called the *product topology* on X.

Theorem 6.4: *Let $(X_i, \mathfrak{T}_i)_{i\in I}$ be a family of topological spaces and $X = \prod_{i\in I} X_i$ be equipped with the product topology. Let $(Y, \mathfrak{T}')$ be another topological space. Then the following statements holds:*

(i) *The collection of sets*

$$\begin{aligned} \mathfrak{S} &:= \{\pi_i^{-1}(V) \mid i \in I, V \in \mathfrak{T}_i\} \\ &= \Big\{U = \prod_{i\in I} U_i \mid U_i \in \mathfrak{T}_i \,\forall\, i \in I,\, \exists\, i_0 \in I : U_i = X_i \,\forall\, i \in I \setminus \{i_0\}\Big\} \end{aligned}$$

is a subbase for $\mathfrak{T}$ *and*

$$\mathfrak{B} = \left\{ U = \prod_{i \in I} U_i \mid U_i \in \mathfrak{T}_i \, \forall \, i \in I : \exists \, J \subseteq I, \#(J) < \infty : U_i = X_i \, \forall \, i \in I \setminus J \right\}$$

is a base for $\mathfrak{T}$.

(ii) $f : (Y, \mathfrak{T}') \to (X, \mathfrak{T})$ *is continuous if and only if* $f_i := \pi_i \circ f : (Y, t) \to (X_i, \mathfrak{T}_i)$ *are continuous for all* $i \in I$.

(iii) *A sequence* $((x_i^{(k)})_{i \in I})_{k \in \mathbb{N}}$ *is convergent in* $(X, \mathfrak{T})$ *to some* $x = (x_i)_{i \in I} \in X$ *if and only if* $\forall \, i \in I : x_i^{(k)} \to x_i$ *in* $(X_i, \mathfrak{T}_i)$.

(iv) *Let* $(X_i, d_i)_{i=1}^N$ *be metric spaces. Then, the product topology on* $X = \prod_{i=1}^n X_i$ *is induced by the metric*

$$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R} \\ ((x_i)_{i=1}^N, (y_i)_{i=1}^N) &\longmapsto \max_{1 \le i \le N} d_i(x_i, y_i), \end{aligned}$$

or by equivalent metrics.

Proof: (i) This follows directly from (Theorem 6.1) and the fact, that products interchange with intersections.

(ii) This stateent is proven in (Theorem 6.1)(ii).

(iii) "$\Rightarrow$": This follows from (Theorem 6.1)(i), because $\pi_i(x^{(n)}) = x_i^{(n)} \, \forall \, i \in I, \forall \, n \in \mathbb{N}$ and $\pi_i(x) = x_i$.
"$\Leftarrow$": Let $U \in \mathfrak{U}(x)$. By (i) and (Lemma 5.5), there is a finite $J = \{j_1, \ldots, j_n\} \subseteq I$ and a base element $B = \prod_{i=1}^n U_i$ with $U_i \subseteq X_i$ open for all $i \in I$ and $U_i = X_i \, \forall \, i \in I \setminus J : x \in B \subseteq U$. This implies, that $\exists \, N \in \mathbb{N} : \forall \, i = 1, \ldots, n \, \forall \, n \ge N : x_{j_i}^{(n)} \in U_{j_i}$, thus $\forall \, n \in \mathbb{N} : x^{(n)} \in B \subseteq U$. This shows, that $x^{(n)} \to x$ as $n \to \infty$.

(iv) The product topology and $\mathfrak{T}_d$ have the base

$$\{B_\varepsilon^d((x_i)_{i=1}^N) \mid (x_i)_{i=1}^N \in X, \varepsilon > 0\} = \left\{ \prod_{i=1}^N B_\varepsilon^{d_i}(x_i) : (x_i)_{i=1}^N, \varepsilon > 0 \right\},$$

so they are the same (for the other metric, show as in Analysis II, that is equivalent to the maximum metric). ■

Definition 6.5: Let $(X, \mathfrak{T})$ be a topological space.

(i) Let A be a quasi-ordered set (that is, there is a relation "$\le$" on A which is reflexive and transitive). A is called *directed* if for all $\alpha, \beta \in A$ there is $\gamma \in A$ such that $\alpha \le \gamma, \beta \le \gamma$.

(ii) A *net in* X is a family $(x_\alpha)_{\alpha \in A}$ with a directed index set $A \neq \{\}$.

For the rest of the definition, let $(x_\alpha)_{\alpha\in A}$ and $(y_i)_{i\in I}$ be nets in X, $x \in X$.

(iii) $(x_\alpha)_{\alpha\in A}$ *converges to* x if $\forall\, U \in \mathfrak{U}(x) : \exists\, \alpha_0 \in A \,\forall\, \alpha \geq \alpha_0 : x_\alpha \in U$.

(iv) $(x_\alpha)_{\alpha\in A}$ *has a cluster point* if $\forall\, U \in \mathfrak{U}(x) : \forall\, \alpha \in A \,\exists\, \beta \in A : \beta \geq \alpha : x_\beta \in U$.

(v) The net $(y_i)_{i\in I}$ is called a *subnet of* $(x_\alpha)_{\alpha\in A}$ if there is a function $\varphi : I \to A, i \to \alpha_i$ such that

 (1) $y_i = x_{\alpha_i} \,\forall\, i \in I$,
 (2) $i, j \in I, i \leq j \Rightarrow \alpha_i \leq \alpha_j$,
 (3) $\forall\, \alpha \in A : \exists\, i \in I : \alpha_i \geq \alpha$.

 We often write $(x_{\alpha_i})_{i\in I}$ for the subnet $(y_i)_{i\in I}$.

Note that $\#(I) > \#(A)$ might occur.

Remark 6.6: (i) Exactly as for sequences, one shows that limits of nets are unique in Hausdorff spaces.

(ii) Let $(X, \mathfrak{T})$ be a topological space, $x \in X$. Then the set of neighbourhoods of x, $\mathfrak{U}(x)$, is directed by "$U \leq V \Leftrightarrow U \supseteq V$" – for $U, V \in \mathfrak{U}(x)$ is a common upper bound.

(iii) Let $a, b \in \mathbb{R}$ with $a < b$ and let

$$\begin{aligned}\mathscr{P} := \big\{\big((p_0, \dots, p_n), (\xi_1, \dots, \xi_n)\big) \mid n \in \mathbb{N},\\ a \leq p_0 < \dots < p_n \leq b, p_{i-1} \leq \xi_i \leq p_i \,\forall\, i = 1, \dots, n\big\}\end{aligned}$$

be directed by

$$\begin{aligned}&\big((p_0, \dots, p_n), (\xi_1, \dots, \xi_n)\big) \leq \big((q_0, \dots, q_m), (\eta_1, \dots, \eta_m)\big) \Leftrightarrow\\ &\exists\, (j_i)_{i=1}^n \in \{1, \dots, m\}^n \,\forall\, i = 1, \dots, n : p_i = q_{i_j}, \xi_i = \eta_k \text{ for some } j_i \leq k \leq j_{i+1}.\end{aligned}$$

If $f : [a, b] \to \mathbb{R}$ is Riemann-integrable, one can show that

$$\int_a^b f(x)\, dx = \lim_{((p_0,\dots,p_n),(\xi_1,\dots,\xi_n))} \sum_{i=1}^{n-1} |p_{i+1} - p_i| f(\xi_{i+1}),$$

in particular, properties like additivity, monotonocity, ... can be derived from more general results.

Theorem 6.7: *Let* $(X, \mathfrak{T})$ *be a topological space,* $(Y, \mathfrak{T}')$ *a topological space,* $A \subseteq X$, $x \in X$ *and* $f : X \to Y$. *Then:*

(i) $\operatorname{cl}(A) = \{x \in X \mid \exists\, (x_i)_{i\in I} \in A^I : x_i \to x\}$,

(ii) f *is continuous in* x *if and only if for any net* $(x_i)_{i\in I} \in X^I$ *with* $x_i \to x$: $f(x_i) \to f(x)$.

(iii) *Let $(X_i, \mathfrak{T}_i)_{i\in I}$ be topological spaces, $X = \prod_{i\in I} X_i$ be equipped with the product topology. A net $(x^{(\alpha)})$ in X (with $x^{(\alpha)} = x^{(\alpha)}_{i\in I}$ for $\alpha \in A$) converges to $x = (x_i)_{i\in I} \in X$ if and only if $x_i^{(\alpha)} \to x_i \,\forall\, \alpha \in A$.*

Proof: (i) "$\supseteq$": Let $x = \lim_{i\in I} x_i$ for a net $(x_i)_{i\in I}$ in A, $U \in \mathfrak{U}(x)$, there is $i_0 \in I \,\forall\, i \geq i_0 : x_i \in U \cap A$, in particular, $U \cap A \neq \{\}$.

"$\subseteq$": Let $x \in \mathrm{cl}(A)$. For $U \in \mathfrak{U}(x)$, choose $x_U \in U \cap A$. Then $(x_U)_{U\in\mathfrak{U}(x)}$ is a net in A. For any $U \in \mathfrak{U}(x)$ and $V \in \mathfrak{U}(x)$ with $V \geq U$: $x_V \in V \subseteq U$, thus $x_U \to x \in A$.

(ii) "$\Rightarrow$": Let $(x_i)_{i\in I}$ be a net in X with $x_i \to x$ and let $V \in \mathfrak{U}(f(x))$. Then, there is $U \in \mathfrak{U}(x) : f(U) \subseteq V$, thus there is $i_0 \in I : \forall\, i \geq i_0 : x_i \in U$ and thus $\forall\, i \geq i_0 : f(x_i) \in f(U) \subseteq V$, hence $f(x_i) \to f(x)$.

"$\Leftarrow$": Let f fulfill the right side and assume, there is $V \in \mathfrak{U}(f(x)) : \forall U \in \mathfrak{U}(x) : f(U) \not\subseteq V$. Then, for $U \in \mathfrak{U}(x)$ choose $x_U \in U$ with $f(x_U) \notin V$. As in (i): $x_U \to x$. But due to $f(x_U) \notin V \,\forall\, U \in \mathfrak{U}(x), f(x_U) \not\to f(x)$.

(iii) "$\Rightarrow$": This follows from (ii) and (Theorem 6.1)(i).

"$\Leftarrow$": Let $U \in \mathfrak{U}(x)$. By (Theorem 6.4)(i) and (Lemma 5.5), there are a finite $J = \{j_1, \ldots, j_n\} \subseteq I$ and a $B = \prod_{i\in I} U_i$ with $U_i \in \mathfrak{T}_i \,\forall\, i \in I$ and $U_i = X_i \,\forall\, i \in I \backslash J$ such that $x \in B \subseteq U$. Then $\forall\, i = 1, \ldots, n : \exists\, \alpha_i \in A : \forall\, \alpha \geq \alpha_i : x_{j_i}^{(\alpha)} \in U_{j_i}$. Because A is directed, we can show via induction, that there is $\alpha_0 \in A : \alpha_i \leq \alpha_0 \,\forall\, i = 1, \ldots, n$ such that $\forall\, \alpha \geq \alpha_0 \,\forall\, i = 1, \ldots n : x_{j_i}^{(\alpha)} \in U_{j_i}$, thus $\forall\, \alpha \geq \alpha_0 : x^{(\alpha)} \in B \subseteq U$. Thus $x^{(\alpha)} \to x$. ■

Theorem 6.8: *Let $(X_n, \mathfrak{T}_n)_{n\in\mathbb{N}}$ be a sequence of metrizable topological spaces (i. e. there are metrics d_i for $i \in \mathbb{N}$) such that $\mathfrak{T} = \mathfrak{T}_{d_i}$). Then, $X = \prod_{n\in\mathbb{N}} X_n$ with the product topology $\mathfrak{T}$ is metrizable with the metric*

$$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R} \\ ((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) &\longmapsto \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}. \end{aligned}$$

Proof: Let $f : (X, \mathfrak{T}_d) \to (X, \mathfrak{T}), x \mapsto x$. We have for a net $(x^{(\alpha)})_{\alpha\in A} = ((x_i^{(\alpha)})_{i\in I})_{\alpha\in A}$ in X and $x = (x_i)_{i\in I} \in X$ it holds

$$x^{(\alpha)} \xrightarrow[\mathfrak{T}]{\alpha} x \Leftrightarrow x_i^{(\alpha)} \xrightarrow[\mathfrak{T}_i]{\alpha} \forall\, i \in \mathbb{N} \Leftrightarrow x^{(\alpha)} \xrightarrow[\mathfrak{T}_d]{\alpha} x$$

By (Theorem 6.7) (ii), f is a homeomorphism, thus $\mathfrak{T}_d = \mathfrak{T}$. ■

7 Compact sets

Definition 7.1: A topological space $(X, \mathfrak{T})$ is called *sequentially compact*, if every sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. $K \subseteq X$ is called sequentially compact, if $(K, \mathfrak{T}|_K)$ is sequentially compact.

Theorem 7.2: *Let (X, d) is a metric space. Then: $(X, \mathfrak{T}_d)$ is compact if and only if $(X, \mathfrak{T}_d)$ is sequentially compact.*

Proof: "$\Rightarrow$": Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. For $m \in \mathbb{N}$, let

$$K_m := \mathrm{cl}(\{x_n \mid n \geq m\}).$$

Then the sequence of sets $(K_m)_{m\in\mathbb{N}}$ has the finite intersection property, so there is $x \in \bigcap_{m\in\mathbb{N}} K_m$. Thus, for all $k \geq 1, n \in \mathbb{N} : B_{1/k}(x) \cap \{x_l \mid l \geq n\} \neq \{\}$. Inductively: there is a monotonly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $x_{n_k} \in B_{1/k}(x) \,\forall\, k \geq 1$. Therefore, $(x_{n_k})_{k\geq 1}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ with $x_{n_k} \to x$ as $k \to \infty$.

"$\Leftarrow$": Let $(U_i)_{i\in I}$ be an open cover for X. Via (Lemma 5.6) (ii), there is a sequence $(i_n)_{n\in\mathbb{N}}$ such that $X = \bigcup_{n\in\mathbb{N}} U_{i_n}$. Assume $\forall\, n \in \mathbb{N}\, \exists\, x_n \in X \setminus U_{i_0} \cup \cdots \cup U_{i_n}$. By assumption, the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ to some $x \in X$. Then $\forall\, l \in \mathbb{N} : x = \lim_{n\to\infty, k\geq l} x_{n_k} \in X \setminus (U_{i_0} \cup \cdots \cup U_{i_l})$, thus $X \setminus (U_{i_0} \cup \cdots \cup U_{i_l}) \subseteq X$ is closed which contradicts $X = \bigcup_{n\in\mathbb{N}} U_{i_n}$. ■

Theorem 7.3: *Let $(X, \mathfrak{T})$ be a topological space. Then X is compact if and only if every net in X has a converging subnet if and only if every net in X has a cluster point.*

Proof: The second equivalence follows immediately from Exercise 33 e).

"$\Rightarrow$": Let $(x_\alpha)_{\alpha\in A}$ be a net in X. For $\beta \in A$, let $K_\beta := \mathrm{cl}\{x_\alpha \mid \alpha \geq \beta\}$. Then $(K_\beta)_{\beta\in A}$ is a family of sets closed in $(X, \mathfrak{T})$ which has the finite intersection property since A is directed. By (Lemma 4.9) (iii) there is $x \in \bigcap_{\beta\in A} K_\beta$. Let $I = \{(U, \alpha) \in \mathfrak{U}(x) \times A \mid x_\alpha \in U\}$. I is non-empty due to $(X, \alpha) \in I$ for all $\alpha \in A$. I can be directed by

$$(U, \alpha) \leq (V, \beta) \quad :\Leftrightarrow \quad U \supseteq V \text{ and } \alpha \leq \beta$$

because: Let $(U, \alpha), (V, \beta) \in I$, then $U \cap V \in \mathfrak{U}(x)$ and there is $\gamma \in A : \alpha, \beta \leq \gamma$. By choice of x there is $\gamma' \in A$ such that $\gamma' \geq \gamma$ and $x_{\gamma'} \in U \cap V$, therefore $(U \cap V, \gamma') \in I$.

For $(U, \alpha) \in I$, let $y_{(U,\alpha)} = x_\alpha$. Then $(y_{(U,\alpha)})_{(U,\alpha)\in I}$ is a subnet of $(x_\alpha)_{\alpha\in A}$. Let any $U \in \mathfrak{U}(x)$ be given and let $\alpha \in A$ with $(U, \alpha) \in I$. Then for all $(V, \beta) \geq (U, \alpha)$ it holds $y_{V,b} = x_\beta \in V \subseteq U$. Hence $y_{(U,\alpha)} \to x$.

"$\Leftarrow$": Let $\{\} \neq \mathfrak{F} \subseteq \mathfrak{P}(X)$ be a collection of non-empty closed subsets of X with finite intersection property. Define

$$\mathfrak{A} := \{F_1 \cap \cdots \cap F_n \mid n \in \mathbb{N}, F_1, \dots F_n \in \mathfrak{F}\}.$$

Because F has the finite intersection property, for all $F \in \mathfrak{A}$ there is $x_F \in F$. Then $(x_F)_{F\in\mathfrak{A}}$ is a net where $\mathfrak{A}$ is directed by $F \leq G \Leftrightarrow F \supseteq G$. By assumption there is a subnet $(x_{F_i})_{i\in I}$ of $(x_F)_{F\in\mathfrak{A}}$, such that $x_{F_i} \to x$ for some $x \in X$. Let $F_0 \in \mathfrak{F}$, then there is $i_0 \in I$ such that $F_{i_0} \geq F_0$ and thus $x = \lim_{i\in I} x_{F_i} = \lim_{i\in I, i\geq i_0} x_{F_i} \in F_0$ since $x_i \in F_i \subseteq F_{i_0}$ for any $i \geq i_0$, $x_{i_0} \in F_{i_0} \subseteq F_0$ and F_0 is closed. ■

Remark 7.4: By passing to subnets (respectively subsequences) repeatedly and using (Theorem 7.3), one can show that finite products of compact (respectively sequentially compact) topological spaces X are compact (respectively sequentially compact) again. With a diagonalization trick, one can even show that countable products of sequentially compact topological spaces are sequentially compact again.

Definition 7.5: Let $\{\} \neq M$ be a partially ordered[1] set and $\{\} \neq C \subseteq M$ a subset.

(i) C is called a *chain* or *totally ordered* if: $x, y \in C : x \leq y \vee y \leq x$.

(ii) An upper bound of C is an element $z \in M$ with $x \leq z \,\forall\, x \in C$.

(iii) A maximal element is an element $z \in M$ such that $(x \in M, z \leq x \Rightarrow x = z)$.

With this, one can show the following Theorem (it is even equivalent to the axiom of choice in ZFC).

Theorem 7.6 (Zorn's Lemma): *Every partially ordered set in which every chain has an upper bound, has a maximal element.*

Theorem 7.7 (Tychonoff's Theorem[2]): *Let $(X_i, \mathfrak{T}_i)_{i\in I}$ be a family of compact topological spaces and let $X = \prod_{i\in I} X_i$ be equipped with the product topology $\mathfrak{T}$. Then $(X, \mathfrak{T})$ is compact.*

Proof: For $J \subseteq I$, let $X_J = \prod_{i\in J} X_i$ be equipped with the product topology. Let $(x^{(\alpha)})_{\alpha\in A} := x$ be a net in X. For $J \subseteq I$, we call an element $g \in X_y$ a partial cluster point $(x)_{\alpha\in A}$, if g is a cluster point of $x|_J = (x^{(\alpha)}|_J)_{\alpha\in A}$.

Then, let

$$\mathfrak{H} := \{g \mid g \in X_J \text{ is partial cluster point of } x, J \subseteq I\}$$

be partially ordered by

$$(g_i)_{i\in J} \leq (h_i)_{i\in K} \quad :\Leftrightarrow \quad J \subseteq K \wedge g_i = h_i \,\forall\, i \in J.$$

[1] There is an antisymmetric quasi-ordering "$\leq$" on M.
[2] Tchonoff's Theorem can be shown to be equivalent to the axiom of choice.

$\mathfrak{H}$ is non-empty since for $i_0 \in I$, $(x_{i_0}^{(\alpha)})_{\alpha \in A}$ has a cluster point $g \in X_{i_0}$ where we identify $X_{i_0} = \prod_{i \in \{i_0} X_i$.

Let $\mathfrak{C} = \{g^{(\lambda)} = (g_i^{(\lambda)})_{i \in J_\lambda} \mid \lambda \in \Lambda\}$ be a chain in $(\mathfrak{H}, \leq)$. Now, let $J = \bigcup_{\lambda \in \Lambda} J_\lambda \subseteq I$ and $G \in \prod_{i \in J} X_i$ by $G_i = g_i^{(\lambda)}$ if $i \in J_\lambda$. This is well-defined: If $i \in J_\lambda$, $i \in J_\mu$, without loss of generality $g^{(\lambda)} \leq g^{(\mu)}$, but then $g_i^{(\lambda)} = g_i^{(\mu)}$.

Next, let $\alpha \in A$, $U \in \mathfrak{U}(G)$. Without loss of generality $U = \prod_{i \in J} U_i$, $U_i \subseteq X_i$ open $\forall\, i \in J$, $\exists\, F \subseteq_{\text{fin}} J : U_i = X_i \,\forall\, i \in J \setminus F$. Now $\forall\, f \in F : \exists\, \lambda_f \in \Lambda : f \in J_{\lambda_f} \Rightarrow \exists\, \lambda_0 \in \Lambda : J_{\lambda_f} \subseteq J_{\lambda_0} \,\forall\, f \in F$, thus $F \subseteq J_{\lambda_0}$. Thus $\exists\, \beta \geq \alpha : \forall\, i \in F : x_i^{(\beta)} \in U_i$, thus $x^{(\beta)} \in U$. Thus, $G \in \mathfrak{H} \Rightarrow G$ is an upper bound for $\mathfrak{C}$. Via Zorns Lemma, there exists a maximal element $g = (g_i)_{i \in J} \in \mathfrak{H}$.

Assume, $J \subsetneq I$. Let $k \in I \setminus J$. Since $g \in \mathfrak{H}$, g is a cluster point of $(x^{(\alpha)}|_J)_{\alpha \in A}$. By Exercise 33 e) we know that there is a subnet $(x^{\alpha+}|_J)_{t \in T}$ of $x|_J$ such that $x^{\alpha_t}|_J \to g$. Because $(X_k, \mathfrak{T}_k)$ is compact, the net $(x_k^{\alpha_t})_{t \in T}$ has a cluster point $p \in X_k$. For $i \in J \cup \{k\}$, let

$$h_i = \begin{cases} y_i & \text{if } i \in J, \\ p & \text{if } i = k. \end{cases}$$

Futhermore let $h = (h_i)_{i \in J \cup \{k\}}$. Let $\alpha \in A$, $U \in \mathfrak{U}(h)$. Without loss of gernerality $U = \prod_{i \in J \cup \{k\}} U_i$ with $U_i \subseteq X_i$ open for all i and $\exists\, F \subseteq_{\text{fin}} J \cup \{k\} : U_i = X_i \,\forall\, i \in J \cup \{k\} \setminus F$. Therefore $\exists\, t_0' \in T : \alpha_{t_0'} \geq \alpha$ and $\forall\, t \geq t_0' : x^{(\alpha_t)}|_J \in \prod_{i \in J} U_i \in \mathfrak{U}(g)$. Thus $\exists\, \alpha_{t_0} \geq \alpha : x_i^{(\alpha_{t_0}} \in U_i \,\forall\, i \in F$. Thus $x^{\alpha_{t_0}}|_{J \cup \{k\}} \in U$. Thus, $h \in \mathfrak{H}$ which is a contradiction.

Therefore $J = I$. Thus, g is already a cluster point. Thus, by (Theorem 7.3), $(X, \mathfrak{T})$ is compact. ∎

Corollary 7.8: *Let $K \subseteq \mathbb{R}^n$, $\mathbb{R}^n$ equipped with $\mathfrak{T}_{\|\cdot\|_\infty}$. Then K is compact if and only if it is closed in $\mathbb{R}^n$ and bounded.*

Proof: "$\Rightarrow$": Let $K \subseteq \mathbb{R}^n$ be compact. Then, via (Theorem 4.10) (ii), $K \subseteq \mathbb{R}^n$ is closed. Since $K \subseteq \bigcup_{n \in \mathbb{N}} B_N(0)$ is an open cover, there are finitely many $i_1, \ldots, i_n$ such that $K \subseteq \bigcup_{j=1}^n B_{N_i}(0)$. Then $K \subseteq B_{\max_{1 \leq j \leq n} i_j}(0)$ and thus bounded.

"$\Leftarrow$": Let $K \subseteq \mathbb{R}^n$ be closed and bounded. Then there is $L \in \mathbb{R}$ such that $K \subseteq B_L(0) = \prod_{i=1}^n [-L, L]$. By Tychonoff, $\prod_{i=1}^n [-L, L]$ is compact. ∎

Definition 7.9: A topological space $(X, \mathfrak{T})$ is called *locally compact*, if every $x \in X$ has a compact neighbourhood $K \in \mathfrak{U}(x)$.

Example 7.10: (i) $(\mathbb{R}^n, \mathfrak{T}_{\|\cdot\|_\infty})$ and $(\mathbb{C}^n, \mathfrak{T}_{\|\cdot\|_\infty})$ are locally compact by (Corollary 7.9). For $x \in \mathbb{R}^n$, $\text{cl}(B_r(x))$ is a compact neighbourhood of x.

(ii) *Differentiable manifolds* (and even more generally *topological manifolds*) are locally compact, because every point has a neighbourhood homeomorphic to an open set in some $\mathbb{R}^n$.

(iii) $R^{\mathbb{N}} = \prod_{n\in\mathbb{N}} \mathbb{R}$ with the product topology (also induced by a metric) is not locally compact. Assume, a point $x \in \mathbb{R}^{\mathbb{N}}$ had a compact neighbourhood K. Then there existed $U = U_0 \times \cdots \times U_N \times \prod_{n=N+1}^{\infty} \mathbb{R}$ with $U_i \subseteq \mathbb{R}$ open for all $0 \leq U_i \leq N$ such that $x \in U \subseteq K$ held. Via (Theorem 4.10), $\mathrm{cl}(U) \subseteq K$ was compact. But then, the image of $\mathrm{cl}(U)$ under the $N+1$-th continuous projection $\pi_{N+1}(\mathrm{cl}(U)) = \mathbb{R}$ was compact, which is a contradiction.

(iv) $(E, \|\cdot\|)$ is locally compact if and only if it is finite-dimensional by Functional Analysis, as $\mathrm{cl}(B_\varepsilon(x))$ is not compact for infinite-dimensional normed spaces.

Theorem 7.11 (Alexandroff-Extension[3]): *Let $(X, \mathfrak{T})$ be a Hausdorff space, $\infty \notin X$, $\widehat{X} = X \cup \{\infty\}$. Then,*

$$\widehat{\mathfrak{T}} = \mathfrak{T} \cup \{\widehat{X} \setminus K \mid K \subseteq X \ \textit{compact}\} \cup \{\widehat{X}\}$$

defines a topology on X such that

(i) *$\widehat{T}|_X = \mathfrak{T}$, $X \subseteq \widehat{X}$ is open,*

(ii) *$(\widehat{X}, \widehat{\mathfrak{T}})$ is compact.*

If $(X, \mathfrak{T})$ is locally compact, $(\widehat{X}, \widehat{\mathfrak{T}})$ is Hausdorff. If X is not compact, then $\mathrm{cl}(X)^{\widehat{X}} = \widehat{X}$.

Proof: First, we want to show that $\widehat{\mathfrak{T}}$ is indeed a topology. That $\{\}, \widehat{X} \in \widehat{\mathfrak{T}}$ holds is clear. Let now $U \in \mathfrak{T}$, $K, L \subseteq X$ compact. Then

$$U \cap (\widehat{X} \setminus K) = U \cap (X \setminus K) \in \mathfrak{T} \subseteq \widehat{\mathfrak{T}} \quad (\widehat{X} \setminus K) \cap (\widehat{X} \setminus L) = \widehat{X} \setminus (K \cup L) \in \mathfrak{T},$$

thus via induction, $\widehat{\mathfrak{T}}$ is closed under finite intersections. Let $(K_i)_{i\in I}$ be a family of compact subsets of X, $U \in \mathfrak{T}$, $K \subseteq X$ compact. Then

$$\bigcup_{i\in I} \widehat{X} \setminus K_i = \widehat{X} \setminus \bigcap_{i\in I} K_i \in \widehat{\mathfrak{T}}$$

holds, because there is $i_0 \in I$ such that $\bigcap_{i\in I} K_i \subseteq K_{i_0}$ and

$$U \cup \widehat{X} \setminus K = \widehat{X} \setminus (K \cap X \setminus U) \in \mathfrak{T},$$

as $K \cap X \setminus U$ is closed and contained in K.

(i) It holds

$$\widehat{\mathfrak{T}}|_X = \{V \cap X \mid V \in \widehat{\mathfrak{T}}\} = \mathfrak{T} \cup \{X \setminus K \mid K \subseteq X \text{ compact}\} \cup \{X\} = \mathfrak{T},$$

and $X \in \mathfrak{T} \subseteq \widehat{\mathfrak{T}}$. Next, let $\widehat{X} = \bigcup_{i\in I} U_i$ be an open cover, then there is $i_0 \in I$ such that $\infty \in U_{i_0}$, thus $U_{i_0} = \widehat{X} \setminus K_{i_0}$ for some $K_{i_0} \subseteq X$ compact. Hence

[3] This theorem is also called *Alexandroff-Compactification* or *One-point-compactification*.

$\bigcup_{i\in I\setminus\{i_0\}}(U_i \cap X)$ is an open cover in $(X, \mathfrak{T})$. Now, there are $i_1, \ldots, i_n \in I$ such that $K \subseteq \bigcup_{j=1}^n (U_{i_j} \cap X)$, therefore $\widehat{X} = K_{i_0} \cup \widehat{X} \setminus K_{i_0} = \bigcup_{j=0}^n U_{i_j}$. Thus, $(\widehat{X}, \widehat{\mathfrak{T}})$ is compact.

Next, let $(X, \mathfrak{T})$ in addition be locally compact and let $x, y \in \widehat{X}, x \neq y$ and without loss of generality, let $y = \infty$. Then $K \in \mathfrak{U}_{\mathfrak{T}}(x) \subseteq \mathfrak{U}_{\widehat{\mathfrak{T}}}(x)$ is compact in $(X, \mathfrak{T})$. Then, K and $\widehat{X} \setminus K$ are disjoint neighbourhoods of x, y.

Now let X be not compact, $U \in \mathfrak{U}(\infty)$. Then, there is $K \subseteq X$ compact such that $\widehat{X} \setminus K \subseteq U$. In particular, $U \cap X \supseteq \widehat{X} \setminus K \cap X = X \setminus K \neq \{\}$. ■

Theorem 7.12: *Let $(X, \mathfrak{T})$ be a Hausdorff space. Then $(X, \mathfrak{T})$ is locally compact if and only if $\forall\, x \in X \,\forall\, U \in \mathfrak{U}(x) \,\exists\, K \in \mathfrak{U}(x)$ compact such that $x \in K \subseteq U$.*

Proof: "$\Leftarrow$": This is clear.

"$\Rightarrow$": Let $x \in X$, $W \in \mathfrak{U}(x)$, without loss of generality let $W \subseteq X$ be open. Then there is $K \in \mathfrak{U}(x)$ compact. Let $A := K \cap X \setminus W \subseteq K$. Then A is closed because X is Hausdorff and via (Theorem 4.10) (i) A is compact.

For any $y \in A$, let $U_y \in \mathfrak{U}(x)$, $V_y \in \mathfrak{U}(x)$ open in X with $U_y \cap V_y = \{\}$. Then $A \subseteq \bigcup_{y\in A} V_y$ is an open cover. Because A is compact, there are $y_1, \ldots, y_n \in A$ such that $A \subseteq \bigcup_{i=1}^n V_{y_i} =: V$. Let $U = \bigcap_{i=1}^n U_{y_i}$, $U \in \mathfrak{U}(x)$ is open in X and $U \cap V = \{\}$.

Let $\{\} \neq K' = U \cap \mathrm{Int}(K) \in \mathfrak{U}(x)$ open (K' is non-empty because $\mathrm{Int}(K) \neq \{\}$) with

$$\begin{aligned} \mathrm{cl}(K') = \mathrm{cl}(U \cap \mathrm{Int}(K)) &\subseteq \mathrm{cl}(U) \cap K \\ &\subseteq X \setminus V \cap K \\ &\subseteq X \setminus A \cap K = (X \setminus K \cup W) \cap K = W \cap K. \end{aligned}$$

Thus $\mathrm{cl}(K') \in \mathfrak{U}(x)$ is compact by (Theorem 4.10) (i) and $\mathrm{cl}(K') \subseteq W$. ■

Corollary 7.13: *Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space, $K \subseteq X$ compact, $U \subseteq X$ open and $K \subseteq U$. Then there exists $V \subseteq X$ open such that $\mathrm{cl}(V)$ is compact and $K \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$.*

Proof: By (Theorem 7.12), for all $x \in K$ there is $V_x \in \mathfrak{U}(x)$ open such that $\mathrm{cl}(V)_x$ compact and $\mathrm{cl}(V)_x \subseteq U$. Therefore there are $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^n V_{x_i} =: V$ is open in X with $\mathrm{cl}(V) = \bigcup_{i=1}^n \mathrm{cl}(V)_{x_i} \subseteq U$ and $\mathrm{cl}(V) = \bigcup_{i=1}^n \mathrm{cl}(V)_{x_i}$ is compact. ■

Definition 7.14: A topological space $(X, \mathfrak{T})$ is called a *Baire space* if for every sequence $(F_n)_{n\in\mathbb{N}}$ of sets closed in X with $\mathrm{Int}(\bigcup_{n\in\mathbb{N}} F_n) \neq \{\}$, there is $n_0 \in \mathbb{N}$, such that $\mathrm{Int}(F_{n_0}) \neq \{\}$.

Theorem 7.15 (Baire's theorem[4]): *Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space. Then $(X, \mathfrak{T})$ is a Baire space.*

[4]Note that this is neither a generalization nor a special case of (Theorem 2.13).

Proof: Let $\mathrm{Int}(\bigcup_{n\in\mathbb{N}} F_n) \neq \{\}$. Assume for all $n \in \mathbb{N}$ it held $\mathrm{Int}(F_n) = \{\}$. Let $x \in \mathrm{Int}(\bigcup_{n\in\mathbb{N}} F_n)$ and $U \in \mathfrak{U}(x)$ open and compact in X, with $x \in U \subseteq$ with $x \in U \subseteq \bigcup_{n\in\mathbb{N}} F_n$. Put $x_0 := x$ and $B_0 := U$. Because $\mathrm{Int}(F_n) \neq \{\}$, it held that $\mathrm{Int}(B_0) \cap X \setminus F_0 \neq \{\}$. Then we could choose $x_1 \in \mathrm{Int}(B_0) \cap X \setminus F_0$ and $B_1 \subseteq \mathrm{Int}(B_0) \cap X \setminus F_0$. We could continue this inductively and get a sequence $(x_n)_{n\in\mathbb{N}}$ in X and a sequence $(B_n)_{n\in\mathbb{N}}$ of compact sets such that for all $n \in \mathbb{N}$ it held $B_n(x) \in \mathfrak{U}(x_n)$ and for all $n \geq 1$ it held $B_n \subseteq B_{n-1} \cap X \setminus F_{n-1}$. By the finite intersection property of $(B_n)_{n\in\mathbb{N}}$, there is

$$x \in \bigcap_{n\in\mathbb{N}} B_n \subseteq \bigcap_{n\geq 1} B_n \subseteq \bigcap_{n\geq 1} X \setminus F_{n-1} = X \setminus \bigcup_{n\in\mathbb{N}} F_n,$$

but $x \in B_0 \subseteq \bigcup_{n\in\mathbb{N}} F_n$ which is a contradiction. ■

8 Separation Theorems

Definition 8.1: A topological space $(X, \mathfrak{T})$ is called *normal* if for all $F, G \subseteq X$ closed, disjoint, there are $U, V \subseteq X$ open, disjoint, with $F \subseteq U$ and $G \subseteq V$.

Remark 8.2: Let X be a normal topological space, $F \subseteq X$ closed, $W \subseteq X$ open with $F \subseteq W$. Because X is normal, there are $U, V \subseteq X$ open, disjoint such that $F \subseteq U$, $X \setminus W \subseteq V$. In particular $F \subseteq U \subseteq \mathrm{cl}(U) \subseteq X \setminus V \subseteq W$, because $X \setminus V$ is closed.

Lemma 8.3: (i) *Compact Hausdorff spaces are normal.*

(ii) *Metric spaces are normal.*

Proof: (i) Let X be a compact Hausdorff space and $F, G \subseteq X$ closed, disjoint. By (Lemma 4.9) F and G are compact. Exactly as in the proof of (Lemma 4.9) (ii) we see that the following holds: $\forall\, x \in F : \exists\, U_x \in \mathfrak{U}(x)$ open in X and $V_x \subseteq X$ open such that $G \subseteq V_x$ and $U_x \cap V_x = \{\}$. Then, $F \subseteq \bigcup_{x \in F} U_x$ is an open cover. Because F is compact, there are $x_1, \ldots, x_n \in F$ such that $F \subseteq \bigcup_{i=1}^n U_{x_i} =: U$. Then $V := \bigcap_{i=1}^n V_{x_i} \subseteq X$ is open with $G \subseteq V$ and

$$U \cap V = \bigcup_{i=1}^{n} \Big(\bigcap_{j=1}^{n} U_{x_i} \cap V_{x_j} \Big) = \bigcup_{i=1}^{n} \Big(U_{x_i} \cap \bigcap_{j=1}^{n} V_{x_j} \Big) = \{\}.$$

(ii) The proof of this part is Exercise 15 d). ■

Theorem 8.4 (Urysohn's Lemma): *Let X be a normal topological space, $F, G \subseteq X$ closed and disjoint. Then there is a continuous function $f : X \to [0,1]$ such that $f|_F = 0$, $f|_G = 1$.*

Proof: Let $F, G \subseteq X$ be closed and disjoint. Put $U_1 := X \setminus G \subseteq X$, then U_1 is open and $F \subseteq U_1$. By (Remark 8.2) there is another set $U_{1/2} \subseteq X$ open such that $F \subseteq U_{\frac{1}{2}} \subseteq \mathrm{cl}(U_{1/2}) \subseteq U_1$. By (Remark 8.2), there are sets $U_{1/4}, U_{3/4} \subseteq X$ open such that

$$F \subseteq U_{\frac{1}{4}} \subseteq \mathrm{cl}(U_{\frac{1}{4}}) \subseteq U_{\frac{1}{2}} \subseteq \mathrm{cl}(U_{\frac{1}{2}}) \subseteq U_{\frac{3}{4}} \subseteq \mathrm{cl}(U_{\frac{3}{4}}) \subseteq U_1.$$

For $n \in \mathbb{N}$, let $D_n = \{\frac{n}{2^n} \mid 1 \leq n \leq 2^n\}$ and let $D = \bigcup_{n \in \mathbb{N}} D_n$. Inductively for $d \in D$, define an open set $U_d \subseteq X$ such that for all $d, e \in D$ with $d < e$, it holds

$$F \subseteq U_d \subseteq \mathrm{cl}(U_d) \subseteq U_e \subseteq \mathrm{cl}(U_e) \subseteq U_1.$$

Now define $f : X \to [0,1]$ via

$$f(x) := \begin{cases} 0 & x \in \bigcap_{d \in D} U_d, \\ \sup\{d \in D \mid x \notin U_d\} & \text{else.} \end{cases}$$

Then we have by definition: $f|_F = 0$ due to $F \subseteq \bigcap_{d \in D} U_d$ and $f|_G = 1$ since $G \subseteq X \setminus U_d$ for all $d \in D$. Next consider the subbase

$$\mathfrak{S} = \{[0,c) \mid 0 < c < 1\} \cup \{(c,1] \mid 0 < c < 1\}$$

of the topology of $[0,1]$. Let $c \in (0,1)$. For $x \in X$, we have

$$f(x) < c \Leftrightarrow \exists\, t < c : x \in U_t,$$

for "$\Rightarrow$" choose $t \in (f(x), c) \cap D$, for "$\Leftarrow$" use $U_t \subseteq U_s \,\forall\, s \geq t$ and therefore $f(x) \leq t < c$. We also have

$$f(x) > c \Leftrightarrow \exists\, t > c : x \notin \mathrm{cl}(U_t),$$

for "$\Leftarrow$" use that $f(x) \geq t > c$ and for "$\Rightarrow$" use that there is $s > c : x \notin U_s$, thus for all $t \in (c,s) \cap D$ it holds $x \in \mathrm{cl}(U_t)$. Hence

$$f^{-1}([0,c)) = \bigcup_{t<c, t \in D} U_t \subseteq X$$

is open as an arbitrary union of open sets and

$$f^{-1}((c,1]) = \bigcup_{t>c, t \in D} X \setminus \mathrm{cl}(U_t) \subseteq X$$

is open as arbitrary union of open sets. By Exercise 22 d) we conclude that f is continuous. ■

Theorem 8.5 (Tietze Extension Theorem): *Let X be a normal topological space, $A \subseteq X$ closed, $[a,b] \subseteq \mathbb{R}$ a perfect interval and $f : (A, \mathfrak{T}|_A) \to [a,b]$ continuous. Then there is a continuous function $F : X \to [a,b]$ with $F|_A = f$.*

Proof: Without loss of generality we may assume $a = -1$ and $b = 1$, since

$$\begin{aligned} \varphi : [a,b] &\longrightarrow [-1,1] \\ t &\longmapsto \frac{2(t-a)}{b-a} - 1 \end{aligned}$$

is a homeomorphism.

Claim 1: For $r > 0$ and $h : A \to [-r,r]$ continuous, there is $H : X \to [-\frac{r}{3}, \frac{r}{3}]$ continuous with $\|h - H\|_A < 2\frac{r}{3}$.

Consider the disjoint subsets $A_- = h^{-1}([-r, -\frac{r}{3}])$, $A_+ = h^{-1}([\frac{r}{3}, r]) \subseteq X$ that also are closed. By Urysohn's Theorem, there is $H : [-\frac{r}{3}, \frac{r}{3}]$ continuous with $H|_{A_-} = -\frac{r}{3}$ and $H|_{A_+} = \frac{r}{3}$, thus

$$\|h - H\|_A \leq 2\frac{r}{3}.$$

Claim 2: There is a sequence $(g_n)_{n \in \mathbb{N} \setminus \{0\}}$ of continuous functions $g_n : X \to \mathbb{R}$ (for $n \geq 1$) such that

(i) $\|g_n\| \leq (\frac{2}{3})^n$ *for all* $n \geq 1$,

(ii) $\|f - \sum_{i=1}^n g_j\|_A \leq (\frac{2}{3})^n$ *for all* $n \geq 1$.

We prove our claim via induction. For $n = 1$ with $h = f$ and $r = 1$, the function $g_1 = H$ from claim 1 meets our conditions. (different target space!).

For the inductive step, let $g_1, \dots, g_n$ be already constructed. Via claim 1 with $r = (\frac{2}{3})^n$ and $h = (f - \sum_{i=1}^n g_i)|_A$, there is a function $g_{n+1} : X \to [-\frac{1}{3}(\frac{2}{3})^n, \frac{1}{3}(\frac{2}{3})^n]$ with

$$\left\| f - \sum_{i=1}^n g_i - g_{n+1} \right\| \leq \left(\frac{2}{3}\right)^{n+1}.$$

Claim 3: The function $F : X \to \mathbb{R}, F(x) := \sum_{j=1}^\infty g_j(x)$ *is well-defined and continuous with* $F|_A = f$.

The well-definedness follows as $|g_j(x)| \leq \|g_j\|_X \leq (\frac{2}{3})^j$ for all $x \in X$ – then the dominated convergence theorem does the job.

Next, for all $x \in X$, we have

$$\left| F(x) - \sum_{j=1}^n g_j(x) \right| \leq \sum_{j=n+1}^\infty |g_j(x)| \leq \sum_{j=n+1}^\infty \left(\frac{2}{3}\right)^j \to 0,$$

thus $\|F - \sum_{j=1}^n g_j\|_X \leq \sum_{j=n+1}^\infty (\frac{2}{3})^j \to 0$ as $n \to \infty$. Via Exercise 39, we conclude that F is continuous.

Furthermore for all $x \in A$: $|f(x) - \sum_{j=1}^n g_j(x)| \leq (\frac{2}{3})^n \to 0$ as $n \to \infty$, thus $f(x) = \sum_{j=1}^\infty g_j(x) = F(x)$ for all $x \in A$. ■

Corollary 8.6: *Let* $(X, \mathfrak{T})$ *be a normal topological space,* $A \subseteq X$ *closed and* $f : (A, \mathfrak{T}|_A) \to \mathbb{R}$ *be continuous. Then there is an extension* $F : X \to \mathbb{R}$ *that is continuous with* $F|_A = f$.

Proof: Consider

$$\begin{aligned} g : A &\longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ x &\longmapsto \arctan(f(x)). \end{aligned}$$

Via Tietzes extension theorem, there is $G_0 : X \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ continuous such that $G_0|_A = g$. By Urysohn's Lemma there is a function $\theta : X \to [0, 1]$ such that $\theta|_{G_0^{-1}(\{-\frac{\pi}{2}, \frac{\pi}{2}\})} = 0$, $\theta|_A = 1$. Let $G = \theta G_0$. Then

$$G : X \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

is continuous with $G|_A = g$, $G(X) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$, thus $F : X \to \mathbb{R}, x \mapsto \tan(G(x))$ is continuous with $F|_A = f$. ■

Definition 8.7: Let X be a topological space, $f : X \to \mathbb{C}$. Then

$$\operatorname{supp}(f) := \operatorname{cl}(\{x \in X \mid f(x) \neq 0\})$$

is called the *support of* f.

Definition 8.8: Let X be a topological space and $(U_\alpha)_{\alpha\in A}$ be an open cover of X. A family $(\Theta_\alpha)_{\alpha\in A}$ of continuous functions $\Theta_\alpha : X \to [0,1]$ (for $\alpha \in A$) is called *continuous partition of unity*, if the following conditions hold:

(i) $\operatorname{supp}(\Theta_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$,

(ii) $(\operatorname{supp}(\Theta_\alpha))_{\alpha\in A}$ is locally finite, i. e. for all $x \in X$ there is $U \in \mathrm{U}(x)$ such that $\{\alpha \in A \mid \operatorname{supp}(\Theta_\alpha \cap U = \{\}\}$ is finite,

(iii) $\sum_{\alpha\in A} \Theta_\alpha(x) = 1$ for all $x \in X$.

Theorem 8.9: *Let $(X, \mathfrak{T})$ be a normal topological space, $U_1, \dots, U_n \subseteq X$ open with $X = \bigcup_{i=1}^n U_i$. Then, there exists a continuous partition of unity $(\Theta)_{i=1}^n$ with respect to $(U_i)_{i=1}^n$.*

Proof: By (Remark 8.2), there is $V_1 \subseteq X$ open such that

$$X \setminus \Big(\bigcup_{i=2}^n U_i\Big) \subseteq V_1 \subseteq \operatorname{cl}(V_1) \subseteq U_1,$$

in particular it holds that $X = V_1 \cup \bigcup_{i=2}^n U_i$. Inductively there are $V_1, \dots, V_n \subseteq X$ open such that $\operatorname{cl}(V_i) \subseteq U_i$ for $1 \leq i \leq n$ and $X = \bigcup_{i=1}^n V_i$.

Analogeously we find $W_1, \dots, W_n \subseteq X$ open such that $\operatorname{cl}(W_i) \subseteq V_i$ for $1 \leq i \leq n$ and $X = \bigcup_{i=1}^n W_i$. By Urysohns Lemma, for $1 \leq i \leq n$ there are $f_i : X \to [0,1]$ continuous with $f|_{\operatorname{cl}(W_i)} = 1$ and $f_i|_{X\setminus V_i} = 0$. Then, for $1 \leq i \leq n$ it holds that $\operatorname{supp}(f_i) \subseteq \operatorname{cl}(V_i) \subseteq U_i$.

Since $\sum_{i=1}^n f_i(x) \geq 1$ for all $x \in X$, for $1 \leq i \leq n$ the functions

$$\begin{aligned} \Theta_i : X &\longrightarrow [0,1] \\ x &\longmapsto \Big(\sum_{i=1}^n f_i(x)\Big)^{-1} f_i(x) \end{aligned}$$

are well-defined, continuous and satisfy $\operatorname{supp}(\Theta_i) \subseteq \operatorname{supp}(f_i) \subseteq U_i$ as well as $\sum_{i=1}^n \Theta_i(x) = 1$ for all $x \in X$. ■

Remark 8.10: If $(X, \mathfrak{T})$ is a normal topological space, $A \subseteq C(X)$ closed under finite sums and quotients and for $F, G \subseteq X$ closed, there is $f \in A$ with $f|_F = 1$, $f|_G = 0$, a careful analysis of the proof of (Theorem 8.9) shows that $(\Theta_i)_{i=1}^n$ can even be chosen in A.

Definition 8.11: A topological space $(X, \mathfrak{T})$ is called *regular* if it is Hausdorff and the following holds:

$$\forall F \subseteq X \text{ closed } \forall x \in X \setminus F : \exists U, V \subseteq X \text{ open, disjoint} : F \subseteq U, x \in V.$$

Theorem 8.12: *Second-countable regular spaces are normal.*

Note that normal spaces are always regular.

Proof: Let $(X, \mathfrak{T})$ be second-countable, regular, $F, G \subseteq X$ closed and disjoint. With the same arguments as in (Remark 8.2), for all $x \in X$ there is $U_x \in \mathfrak{U}(x)$, U_x open such that $x \in U_x \subseteq \mathrm{cl}(U_x) \subseteq X \setminus G$. Analogously for all $x \in G$ there is $V_x \in \mathfrak{U}(x)$ open in X such that $\mathrm{cl}(V_x) \cap F = \{\}$. By Lindelöfs Theorem there are sequences $(U_n)_{n \in \mathbb{N}}$ in $\{U_x \mid x \in F\}$, $(V_n)_{n \in \mathbb{N}}$ in $\{V_x \mid x \in G\}$ such that $F \subseteq \bigcup_{n \in \mathbb{N}} U_n$, $G \subseteq \bigcup_{n \in \mathbb{N}} V_n$.

For all $n \in \mathbb{N}$, let

$$U'_n = U_n \cap \bigcap_{i=0}^{n} X \setminus \mathrm{cl}(V_i), \qquad V'_n = V_n \cap \bigcap_{i=0}^{n} X \setminus \mathrm{cl}(U_i).$$

Then, let $U = \bigcup_{n \in \mathbb{N}} U'_n$, $V = \bigcup_{n \in \mathbb{N}} V'_n \subseteq X$, U, V then are open in X with $F \subseteq U$, $G \subseteq V$ by construction.

Assume there was $x \in U \cap V$. If there was $x \in U \cap V$, there were $n, m \in \mathbb{N}$ such that it held $x \in U'_n, x \in V'_m$ held. Without loss of generality we could assume $n \leq m$. It then held that $x \in U'_n \subseteq U_n$ and $x \in V'_m \subseteq \bigcap_{i=0}^{n} X \setminus \mathrm{cl}(U_i) \subseteq X \setminus \mathrm{cl}(U_n)$, which is a contradiction, thus $U \cap V = \{\}$. ■

Lemma 8.13: *Locally compact Hausdorff spaces are regular.*

Proof: Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space, $F \subseteq X$ closed and $(\hat{X}, \hat{\mathfrak{T}})$ the one-point compactification of X. As $(\hat{X}, \hat{\mathfrak{T}})$ is Hausdorff, $\{x\}$ and $F \cup \{\infty\} \subseteq \hat{X}$ are closed. By (Lemma 8.3) (i), there are $\hat{U}, \hat{V} \subseteq \hat{X}$ open, disjoint such that $x \in \hat{U}, F \cup \{\infty\} \subseteq \hat{V}$. Then $U = \hat{U} \cap X$, $V = \hat{V} \cap X$ are open, disjoint and it holds $x \in U, F \subseteq V$. ■

Theorem 8.14 (Urysohn's metrization theorem[1]): *Let $(X, \mathfrak{T})$ be a second-countable, regular space. Then there is a homeomorphism $f : X \to f(X) \subseteq \mathbb{R}^{\mathbb{N}}$. In particular $(X, \mathfrak{T})$ is metrizable.*

Proof: Let $\mathfrak{B}$ be a (at most) countable base for $\mathfrak{T}$. Then, the set

$$\mathfrak{M} = \{(B_0, B_1) \in \mathfrak{B}^2 \mid \mathrm{cl}(B_0) \subseteq B_1\}$$

[1] In german, the name "Urysohns Einbettungssatz" which translates to Urysohns embedding theorem, is also used.

is also at most countable, let

$$\begin{aligned}\varphi : \mathbb{N} &\longrightarrow \mathfrak{M}\\ n &\longmapsto (B_0^{(n)}, B_1^{(n)})\end{aligned}$$

be a surjection. By Urysohn's Lemma and (Theorem 8.12), for all $n \in \mathbb{N}$ there is $f_n : X \to [0,1]$ continuous with $f_n|_{\mathrm{cl}(B_0^{(n)})} = 1$, $f_n|_{X \setminus B_1^{(n)}} = 0$. Let

$$\begin{aligned}f' : X &\longrightarrow \mathbb{R}^{\mathbb{N}}\\ x &\longmapsto (f_n(x))_{n \in \mathbb{N}},\end{aligned}$$

then, since $(\pi_n \circ f')$"="f_n (target space!) for all $n \in \mathbb{N}$, f' is continuous. It remains to be shown, that for a net $(x_\alpha)_{\alpha \in A}$ in X and $x \in X$ such that $f'(x_\alpha) \to f'(x)$, it holds that $x_\alpha \to x$. Let $U \in \mathfrak{U}(x)$. Via (Lemma 5.5), there is $B_1 \in \mathfrak{B}$ such that $x \in B_1 \subseteq U$. As X is regular, there is $V \subseteq X$ open such that $x \in V \subseteq \mathrm{cl}(V) \subseteq V_1$, thus by (Lemma 5.5) there is $B_0 \in \mathfrak{B}$ such that $x \in B_0 \subseteq \mathrm{cl}(B_0) \subseteq \mathrm{cl}(V) \subseteq B_1$. Let $n \in \mathbb{N}$ be such that $\varphi(n) = (B_0, B_1)$. Then $f_n(x_\alpha) \to f_n(x) = 1$, thus there is $x_0 \in A$ such that for all $\alpha \geq \alpha_0$ it holds that $f_n(x_\alpha) \in (\frac{1}{2}, \frac{3}{2})$. Then for all $\alpha \geq \alpha_0$ it holds that $x_\alpha \in B_1 \subseteq U$. ■

Lemma 8.15: (i) *Subspaces and products of Hausdorff spaces are Hausdorff,*

(ii) *Subspaces and products of regular spaces are regular.*

Proof: (i) Let $(X, \mathfrak{T})$ be a Hausdorff space, $Y \subseteq X$ and $x, y \in Y$ with $x \neq y$. Then there are $U, V \subseteq X$ open, disjoint such that $x \in U, y \in V$. Then$U \cap Y, V \cap Y \subseteq Y$ are open, disjoint and $x \in U \cap Y, y \in V \cap Y$. That products of Hausdorff spaces are Hausdorff was shown in Exercise 30 (a).

(ii) Let $(X, \mathfrak{T})$ be regular, $Y \subseteq X$, $F \subseteq Y$ closed and $x \in Y \setminus F$. Via (Lemma 4.9) (i), there is $F' \subseteq X$ closed such that $F = F' \cap Y$. As $x \in X \setminus F'$, there are $U', V' \subseteq X$ open, disjoint such that $x \in U'$, $F' \subseteq V'$. Then $U = U' \cap Y$, $V = V' \cap Y$ are open, disjoint such that $x \in U$, $F = F' \cap Y \subseteq V' \cap Y = V$.

Let $(X_i, \mathfrak{T}_i)_{i \in I}$ be a family of regular spaces, $X = \prod_{i \in I} X_i$ equipped with the product topology. Let $F \subseteq X$ be closed, $x = (x_i)_{i \in I} \in X \setminus F$. Then there is $J \subseteq_{\text{fin}} I$ and $U_i \subseteq X_i$ open for $i \in I$ such that $U_i = X_i$ for all $i \in I \setminus J$ and $x \in \prod_{i \in I} U_i \subseteq X \setminus F$. Because X is normal, for all $i \in J$ there is $V_i \subseteq X_i$ open such that $x_i \in V_i \subseteq \mathrm{cl}(V_i) \subseteq U_i$. Let now

$$V = X \setminus \Big(\bigcap_{i \in J} \pi_i^{-1}(\mathrm{cl}(V_i)) \Big) \subseteq X, \qquad U = \bigcap_{i \in J} \pi_i^{-1}(V_i) \subseteq X.$$

Then V, U are open and $U \cap V = \{\}$, furthermore $x \in U$ by the choice of V_i and $F \subseteq X \setminus \prod_{i \in I} U_i \subseteq X \setminus \bigcap_{i \in J} \pi_i^{-1}(\mathrm{cl}(V_i))$. ■

Remark 8.16: Subspacs and (even finite) products of normal spaces need not be normal; refer to Munkres: Topology. A first course, Ex 2 in paragraph 4.2.

Corollary 8.17: *For a topological space $(X, \mathfrak{T})$, the following is equivalent:*

(i) *X is a metrizable and second-countable,*

(ii) *X is metrizable and separable,*

(iii) *X is regular and second-countable,*

(iv) *X is homemorphic to a subspace of $\mathbb{R}^{\mathbb{N}}$.*

Proof: "(i) $\Rightarrow$ (ii)" is (Lemma 5.6) (iii), "(ii) $\Rightarrow$ (iii)" is (Lemma 8.3) and (Lemma 5.6) (iii), "(iii) $\Rightarrow$ (iv)" is (Theorem 8.14) and finally for "(iv) $\Rightarrow$ (i)": Via (Theorem 6.8), $\mathbb{R}^{\mathbb{N}}$ is metrizable and therefore X is metrizable; via (Example 5.9) (ii) $\mathbb{R}^{\mathbb{N}}$ is separable, therefore by (Corollory 5.7) X is separable and thus by (Lemma 5.6) (iii) X then is second-countable. ■

Remark 8.18: There are several other notions of separability... Diagrams missing.