UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercises for the lecture topology

Winter term 2017/18

Sheet 11 Deadline: Wednesday, 17/1/2018, just before the lecture

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Exercise 47 (2+2=4 points)

(In locally compact Hausdorff spaces, finite open covers of compact subsets still admit a continuous partition of unity)

Let X be a locally compact Hausdorff space and let $K \subset U_1 \cup \ldots \cup U_n$ be an open cover of a compact subset $K \subset X$. Show that the following assertions hold:

(a) There are open sets $V_1, \ldots, V_n \subset X$ such that $K \subset V_1 \cup \ldots \cup V_n$, $\overline{V_i} \subset U_i$ and $\overline{V_i}$ is compact for all $1 \leq i \leq n$.

(b) There are continuous functions $f_1, \ldots, f_n : X \to [0,1]$ with compact supports such that $\operatorname{supp}(f_i) \subset U_i$ $(i = 1, \ldots, n)$ and $\sum\limits_{i=1}^{n} f_i(x) = 1$ holds for all $x \in K$.

(Hint: A continuous function $\Theta : X \to [0,1]$ with $\Theta|_K = 1$ and supp$(\Theta) \subseteq V_1 \cup \ldots \cup V_n$ where $V_1, \ldots, V_n \subseteq X$ are sets as in a) could be useful.)

A second-countable topological Hausdorff space is called topological m-manifold *if every point $x \in X$ has a neighbourhood which is homeomorphic to an open subset of the Euclidean space $(\mathbb{R}^m, \tau_{\|\cdot\|})$.*

Exercise 48 (4 points)

(Compact topological manifolds can be embedded into an Euclidean space)

Let X be a compact topological m-manifold. Show that there is a natural number N and a topological embedding (= homeomorphism onto its image) $f : X \to \mathbb{R}^N$.

(Hint: Choose an open cover $(U_i)_{i=1}^n$ of X such that there are homeomorphisms $g_i : U_i \to V_i \subset \mathbb{R}^m$ and use a continuous partition of unity with respect to $(U_i)_{i=1}^n$ to construct a function $f : X \to \mathbb{R}^{n+nm}$ which is injective and continuous.)

(please turn the page)

A topological Hausdorff space X is called completely regular *if for every closed set $F \subset X$ and every $x \in X \setminus F$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f|_F \equiv 1$ hold.*

Exercise 49 (1+1+2=4 points)

(Characterisation of completely regular spaces)

Let X be a topological Hausdorff space. Show that:

(a) X completely regular $\Rightarrow$ X regular.

(b) If X is completely regular, so is every subspace $Y \subset X$ (equipped with the relative topology).

(c) X is completely regular if and only if X is homeomorphic to a subspace of a compact Hausdorff space.
(*Hint: Consider the set*

$$\mathcal{C} = \{f : X \to [0,1];\ f \text{ continuous}\}$$

and the function $j : X \to \prod_{\mathcal{C}} [0,1],\ x \mapsto (f(x))_{f \in \mathcal{C}}$.)

Exercise 50 (4 points)

(When is a compact Hausdorff space metrizable?)

Show that a compact Hausdorff space X is metrizable if and only if it is second-countable.