UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercises for the lecture topology
Winter term 2017/18

Sheet 12 Deadline: Wednesday, 24/1/2018, just before the lecture

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Exercise 51 **(2+2=4 points)**

((Path-)Connectedness is preserved by continuous functions; Path-Connectedness implies connectedness)

Let (X, τ), (Y, t) be topological spaces and let $f : X \to Y$ be continuous. Show that:

(a) If X is connected, so is $f(X)$.

(b) If X is path-connected, so is $f(X)$.

(c) If X is path-connected, X is connected.

Exercise 52 **(4 points)**

(Are interior, closure and boundary of connected subsets connected?)

Let (X, τ) be a topological space and $A \subseteq X$. Which of the following six implications hold? (Proof or counterexample)

(a) A connected $\Leftrightarrow$ $\mathrm{Int}(A)$ connected,

(b) A connected $\Leftrightarrow$ $\overline{A}$ connected,

(c) A connected $\Leftrightarrow$ ∂A connected.

Exercise 53 **(1+1+2 = 4 points)**

(Products of (path-)connected spaces)

Let $(X_i, \tau_i)_{i \in I}$ be a family of topological spaces and let $X = \prod_{i \in I} X_i$ be equipped with the product topology. Show that:

(a) X is path-connected if and only if every X_i $(i \in I)$ is path-connected.

(b) For a point $a = (a_i)_{i \in I}$, the set $\{x = (x_i)_{i \in I}; x_i = a_i \text{ for almost every } i \in I\} \subseteq X$ is dense.

(c) X is connected if and only if every X_i $(i \in I)$ is connected.

(please turn the page)

Exercise 54 **(1,5+2,5=4 points)**

Let (X, τ) be a topological space. Show that

(a) For every subset $A \subseteq X$ and every conneceted subset $Y \subseteq X$, we have

$$Y \cap A \neq \varnothing \text{ and } Y \cap (X \setminus A) \neq \varnothing \quad \Rightarrow \quad Y \cap \partial A \neq \varnothing.$$

(b) The following assertions are equivalent:

(i) X is connected,

(ii) For all $\emptyset \neq A \subsetneq X$, we have $\partial A \neq \emptyset$,

(iii) There is no subset $\emptyset \neq A \subsetneq X$ such that $A \subseteq X$ is open and closed.

Subsets A, B of a topological space are called seperated*if $A \cap \overline{B} = \overline{A} \cap B = \varnothing$ holds.*

Exercise 55* **(4* points)**

(Can one characterize when unions of connected subspaces are connected?)

Sei X be a topological space and let $A, B \subset X$ be non-empty, connected subsets of X. Show that the following equivalence holds:

$$A \cup B \text{ is connected} \quad \Leftrightarrow \quad A, B \text{ are not separated.}$$
