UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercises for the lecture topology
Winter term 2017/18

Sheet 13 Deadline: Wednesday, 31/1/2018, just before the lecture

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This additional exercise sheet can serve you as a mock exam. The exercises are approximately as difficult as is to be expected in an exam. Only exercise 56 b) could be a bit harder than a typical exercise in an exam. The choice of topics will also be similar in the exam, but of course not exactly the same.

Exercise 56* **(2+6 = 8 points)**

(a) Let (X, d) be a metric space and let $f : X \to (\mathbb{R}, \tau_{|\cdot|})$, $g : X \to (\mathbb{R}, \tau_{|\cdot|})$ be uniformly continuous functions. Show that the function $h : X \to (\mathbb{R}^2, \tau_{\|\cdot\|_1}), x \mapsto (f(x), g(x))$ is also uniformly continuous.

(b) Let (X, d) be a metric space. Show that (X, d) is complete if and only if every sequence $(x_n)_{n\in\mathbb{N}}$ in X for which the series $\sum_{n=0}^{\infty} d(x_n, x_{n+1})$ converges, is convergent itself.
(*Hinweis: For one implication, it might be prudent to pass from an arbitrary Cauchy sequence to an approbriate subsequence.*)

Exercise 57* **(6* points)**

Let (K, d) be a compact metric space, $\varphi : K \to K$ a function such that

$$d(\varphi(x), \varphi(y)) < d(x, y)$$

holds for all $x, y \in K$ with $x \neq y$. Show that there is exactly one $x_0 \in K$ such that $\varphi(x_0) = x_0$.

Exercise 58* **(2*+4*+2*+2*+3*= 13* points)**

(a) Let (X, τ) be a topological space and let $A \subseteq X$. Show that A is the intersection of a set closed in (X, τ) and a set open in (X, τ) if and only if for every $x \in A$, there is a neighbourhood $U_x \in \mathcal{U}(x)$ such that $U_x \cap A \subseteq U_x$ is closed.

(b) Let (X, τ) be a topological Hausdorff space and let $F \subseteq X$ be finite. Show that $F \subseteq X$ is closed.

(please turn the page)

(c) Let (X, τ) be a topological space and let $A, B \subseteq X$ be compact subsets. Show that $A \cup B$ is compact in (X, τ).

(d) Let (X, τ) be a topological Hausdorff space and let $A, B \subseteq X$ be compact subsets. Show that $A \cap B$ is compact in (X, τ).

(e) Let (X, τ), (Y, t) be topological spaces, let $\mathcal{B} \subseteq \tau$ be a base for τ and let $f : X \to Y$ be a surjective, continuous and open map (open means that f maps open sets to open sets). Show that $f(\mathcal{B})$ is a base for t.

Exercise 59* **(4*+4* = 8* points)**

(a) Let (X, τ) be a topological space. Show that (X, τ) is Hausdorff if and only if for every $x \in X$, we have $\{x\} = \bigcap_{U \in \mathcal{U}(x)} \overline{U}$.

(b) Let (X, τ) be a topological space such that for every $x \in X$, there is a continuous function $f : X \to (\mathbb{R}, \tau_{|\cdot|})$ with $f^{-1}(\{0\}) = \{x\}$. Show that (X, τ) is Hausdorff.

Exercise 60* **(4* points)**

Let (X, τ) be a compact topological space, (Y, t) a topological space and let the cartesian product $X \times Y$ be equipped with the product topology. Show that the projection $\pi_2 : X \times Y \to Y, (x, y) \mapsto y$ is closed (i.e. it maps closed sets to closed sets).

Exercise 61* **(8* points)**

Let (X, τ) be a topological space, (Y, t) a compact Hausdorff space and let the cartesian product $X \times Y$ be equipped with the product topology. Show that the function $f : X \to Y$ is continuous if and only if its graph $\Gamma_f = \{(x, f(x)); x \in X\} \subseteq X \times Y$ is closed.

(*Hinweis: For the implication* $\Leftarrow$, *use a proof by contradiction.*)

Exercise 62* **(6* points)**

Let (X, τ) be a topological space such that $|X| < \infty$ and such that for all $x, y \in X$ with $x \neq X$, there are neighbourhoods $U \in \mathcal{U}(x)$, $V \in \mathcal{U}(y)$ such that $y \notin U$, $x \notin V$. Show that we have $\tau = \mathcal{P}(X)$ in this case.

Exercise 63* **(4*+3* = 7* points)**

(a) Let (X, τ) be a connected topological space and let $f : X \to \mathbb{R}$ be a locally constant function, i.e.

$$\forall x \in X : \exists U \in \mathcal{U}(x) : \exists c \in \mathbb{R} : \forall y \in U : f(y) = c.$$

Show that f is constant.

(b) Find a topological space (X, τ) and a locally constant function $f : X \to \mathbb{R}$ which is not constant.
