UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercises for the lecture topology

Winter term 2017/18

Sheet 1 Deadline: Wednesday, 25/10/2017, just before the lecture

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***Reminder**: A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is called a Cauchy seqence if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ holds for all $n, m \geq N$. It is called convergent to some $x \in X$ if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ holds for all $n \geq N$.*

***Erinnerung**: A subset $A \subseteq X$ of a metric space (X,d) is called bounded if there is $M > 0$ such that $d(x,y) \leq M$ holds for all $x, y \in A$, i.e if $\{d(x,y); x,y \in X\} \subseteq \mathbb{R}$ is bounded. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called bounded if the set $\{x_n, n \in \mathbb{N}\} \subseteq X$ is bounded.*

Exercise 1 (1+1+2 = 4 points)

Let (X,d) be a metric space. Show that

(a) Cauchy sequences (X,d) are bounded.

(b) Convergent sequences in (X,d) are Cauchy sequences.

(c) If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d) which has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges in (X,d) to some $x \in X$, then we also have $\lim_{n\to\infty} x_n = x$ in (X,d).

***Erinnerung**: Let $f : X \to Y$ be a mapping between metric spaces (X, d_X) and (Y, d_Y).*
The function f is called sequentially continuous if for every sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d) with $\lim_{n\to\infty} x_n = x$ for some $x \in X$, we have $\lim_{n\to\infty} f(x_n) = f(x)$.
We call f uniformly continuous if for every $\epsilon > 0$ there is $\delta > 0$ such that for $x, y \in X$ with $d_X(x,y) < \delta$, we already have $d_Y(f(x), f(x)) < \epsilon$.

Exercise 2 (1+1+2+2* = 4+2* points)

Let (X, d_X), (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function.

(a) Let f be uniformly continuous. Show: If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d_X), then $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y, d_Y).

(b) Show that the assumption that f is uniformly continuous in (a) can not be replaced by sequential continuity.

(c) Now let f be such that it maps Cauchy sequences to Cauchy sequences. Show that f does not need to be uniformly continuous.

(d) * Again, let f be such that it maps Cauchy sequences to Cauchy sequences. Show that f is sequentially continuous.

(bitte wenden)

Exercise 3 **($2+2+2^* = 4+2^*$ points)**

Let (X,d) be a metric space, $a \in X$ and $r > 0$.

(a) Show that $\partial B_r(a) \subseteq \{x \in X, d(x,a) = r\}$ and $\overline{B_r(a)} \subseteq \overline{B_r}(a)$ hold.

(b) More particularly, let $(X, \|\cdot\|)$ be a normed space and $d : X \times X \to \mathbb{R}, d(x,y) = \|x-y\|$ the induced metric. Show that we even have $\partial B_r(a) = \{x \in X, d(x,a) = r\}$ and $\overline{B_r(a)} = \overline{B_r}(a)$ in this case.

(c) * Provide an example for a metric space (X,d), an element $a \in X$ and $r > 0$ such that

$$\{\} \neq \partial B_r(a) \neq \{x \in X, d(x,a) = r\}$$

holds.

Exercise 4 **(4x0,5 + 0,5 + 1 + 0,5 = 4 points)**

(a) Provide examples to show that the following identities do not even hold for subsets $A, B \subseteq (\mathbb{R}, d_{|\cdot|})$ respectively a family $(A_i)_{i\in I}$ of subsets of $(\mathbb{R}, d_{|\cdot|})$:

  (i) $\overline{A \cap B} = \overline{A} \cap \overline{B}$,

  (ii) $\mathrm{Int}(A \cup B) = \mathrm{Int}(A) \cup \mathrm{Int}(B)$,

  (iii) $\partial(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \partial A_i$,

  (iv) $\partial A = \partial \overline{A}$.

(b) Let (X,d) be a metric space and let $A \subseteq X$. Show the following identities or provide a counterexample:

  (i) $\mathrm{Int}(\overline{A}) = A$,

  (ii) $\mathrm{Int}(\overline{\mathrm{Int}(\overline{A})}) = \mathrm{Int}\overline{A}$,

  (iii) $\overline{\mathrm{Int}(A)} = A$.

You can hand in exercise sheets in groups of up to 3 students. To be allowed to write the exam, you have to achieve at least 50 % of all achievable points in the exercise.

You can find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre