UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercises for the lecture topology
Winter term 2017/18

Sheet 2 Deadline: Wednesday, 8/11/2017, just before the lecture

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Exercise 5 **(1+1+0,5+0,5+1 = 4 points)**

(a) Let (X, d) be a metric space and let $\emptyset \neq Y \subset X$ be equipped with the relative metric d_Y. Show that:

(i) A subset $U \subseteq Y$ is open in (Y, d_Y) if and only if there is a set V open in (X, d) such that $U = V \cap Y$.

(ii) A subset $A \subseteq Y$ is closed in (Y, d_Y) if and only if there is a set B closed in (X, d) such that $A = B \cap Y$.

(b) Give an example of a metric space (X, d) and subsets $\emptyset \neq A \subsetneq Y \subset X$ such that:

(i) A is open in (Y, d_Y) but not open in (X, d).

(ii) A is closed in (Y, d_Y) but not closed in (X, d).

(iii) A is open and closed in (Y, d_Y).

Exercise 6 **(1,5+1,5+1=4 points)**

Let (X, d) be a complete metric space and let $A : X \to X$ be a function such that there is a constant $\theta \in [0, 1)$ with

$$d(Ax, Ay) \leq \theta d(x, y) \quad (x, y \in X).$$

Given $x_0 \in X$, let the sequence $(x_n)_n$ in X be defined inductively by $x_{n+1} = Ax_n$ $(n \in \mathbb{N})$. Show that:

(a) For all $n, k \in \mathbb{N}$ the inequality $d(x_{n+k}, x_n) \leq \left(\sum_{j=n}^{n+k-1} \theta^j \right) d(x_1, x_0)$ holds.

(b) The sequence $(x_n)_n$ converges to an $a \in X$ with $A(a) = a$.

(c) The fix point a of A is unique, that is if $\tilde{a} \in X$ is further point with $A(\tilde{a}) = \tilde{a}$, we can deduce $\tilde{a} = a$.

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Exercise 7 **(1+1+2+1*+2* = 4 + 3* points)**

Let (X, d) a metric space.

(a) Let $f : [0, \infty) \to [0, \infty)$ be strictly increasing and subadditive (i.e we have $f(s+t) \leq f(s) + f(t)$ for all $s, t \in \mathbb{R}$) with $f(0) = 0$. Show that $f \circ d$ defines a metric on X.

(b) Show that $\tilde{d} : X \times X \to \mathbb{R}, \tilde{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$ defines a metric on X.

(c) Show that (X, d) and $(X, \tilde{d})$ have the same open sets.

(d) * Find a set $X \neq \emptyset$ and metrics d, $\tilde{d}$ on X such that (X, d) and $(X, \tilde{d})$ have the same open sets but not the same bounded sets.

(e) * Let $Y = \{\frac{1}{n}; n \in \mathbb{N}^*\}$ be equipped with the discrete metric d respectively the relative metric d_Y of $(\mathbb{R}, d_{|\cdot|})$. Show: (Y, d) and (Y, d_Y) have the same open sets but (Y, d_Y) is not complete.

Exercise 8 **(4 points)**

Let (X_1, d_1), (X_2, d_2) be complete metric spaces, let (X, d) be a further metric space and let $i_1 : X \to X_1$, $i_2 : X \to X_2$ be isometries with dense image. Show: There is a unique isometric bijection $\Phi : X_1 \to X_2$ with $i_2 = \Phi \circ i_1$.

Exercise 9* **(2*+2*=4* points)**

Let (X, d) be a metric space and let $x_0 \in X$ be a given point. For $x \in X$, define the function $f_x : X \to \mathbb{R}$ by

$$f_x(t) = d(x, t) - d(x_0, t).$$

Show:

(a) $j : X \to \ell^\infty(X),\ j(x) = f_x$ is an isometry between metric spaces.

(b) Give a different proof for the existence of the completion of X (i.e. a proof which does not use an equivalence relation).

Exercise 10* **(2*+2*=4* points)**

Let $((X_n, d_n))_{n \in \mathbb{N}}$ be a sequence of metric spaces and let $X = \prod_{n=0}^{\infty} X_n$ be the Cartesian product of the sets X_n.

(a) Show that

$$d : X \times X \to \mathbb{R},\ d((x_n)_n, (y_n)_n) = \sum_{n=0}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

defines a metric on X.

(b) Show that the metric space (X, d) is complete if and only if all the spaces (X_n, d_n) are complete.

You can find the exercise sheets on our homepage:

http://www.math.uni-sb.de/ag/eschmeier/lehre