UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercises for the lecture topology
Winter term 2017/18

Sheet 4 Deadline: Wednesday, 22/11/2017, just before the lecture

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For a metric space (X, d) and $\varnothing \neq A \subset X$, we call

$$d(x, A) = \inf\{d(x, y);\ y \in A\}$$

the distance *of an element $x \in X$ to the set A.*

Exercise 15 **($1{+}1{+}1{+}1{+}1^* = 4 + 1^*$ points)**
Let (X, d) be a metric space, let $M \subseteq X$ be a nonempty subset and let $A, B \subseteq X$ be closed, nonempty subsets. Show that:

(a) $d_M : X \to \mathbb{R}, x \mapsto d(x, M)$ is continuous,

(b) $\overline{M} = \{x \in X, d(x, M) = 0\}$,

(c) If A and B are disjoint, there is a continuous function $f : X \to [0, 1]$ such that we have for all $x \in X$:

$$f(x) = 0 \Leftrightarrow x \in A \quad \text{und} \quad f(x) = 1 \Leftrightarrow x \in B,$$

(d) If A and B are disjoint, there are disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$,

(e) * There is a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets $U_n \subseteq X$ $(n \in \mathbb{N})$ such that

$$A = \bigcap_{n\in\mathbb{N}} U_n.$$

Exercise 16 **(4 points)**
Let $X \neq \varnothing$ be a set. Show under which conditions the following collections of subsets $\tau_i \subset \mathcal{P}(X)$ $(i = 1, 2, 3)$ are topologies on X and under which conditions they are Hausdorff:

$$\tau_1 = \{U \subseteq X; X \setminus U \text{ is finite}\} \cup \{\emptyset\},$$
$$\tau_2 = \{U \subseteq X; X \setminus U \text{ is countable}\} \cup \{\emptyset\},$$
$$\tau_3 = \{U \subseteq X; X \setminus U \text{ is infinite}\} \cup \{\emptyset\} \cup \{X\}.$$

Exercise 17 **(4 points)**
Let (X, τ_X), (Y, τ_Y) be topological spaces, $f : X \to Y$ a function and $(U_i)_{i\in I}$ be a family of open sets $U_i \subseteq X$ $(i \in I)$ such that $X = \bigcup_{i\in I} U_i$. Show that:

$$f \text{ is continuous} \quad \Leftrightarrow \quad \forall i \in I : \quad f|_{U_i} : (U_i, \tau_X|_{U_i}) \to Y \text{ is continuous}$$

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Exercise 18 **(1+1+1+1 = 4 points)**

Let (X, τ_X), (Y, τ_Y) be topological spaces and $f : X \to Y$ a function. Show that:

(a) f is continuous if and only if for all $V \subseteq Y$, we have $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$,

(b) f is continuous if and only if for all $V \subseteq Y$, we have $f^{-1}(\mathrm{Int}(V)) \subseteq \mathrm{Int}(f^{-1}(V))$,

(c) f is continuous if and only if for all $U \subseteq X$, we have $f(\overline{U}) \subseteq \overline{f(U)}$.

(d) Now, let f be bijective in addition. Show that f is a homeomorphism if and only if for all $U \subseteq X$, we have: $f(\overline{U}) = \overline{f(U)}$.

Exercise 19* **(3*+1*=4* points)**

Let $\mathbb{R}$ be equipped with $d = d_{|\cdot|}$, $f : \mathbb{R} \to \mathbb{R}$ be a functions and for $n \in \mathbb{N}^*$ let

$$U_n = \bigcup \left(U;\ U \subset \mathbb{R} \text{ open with } \mathrm{diam} f(U) < \frac{1}{n} \right).$$

Show that:

(a) $\bigcap\limits_{n \in \mathbb{N}} U_n$ is the set of those real numbers in which f is continuous.

(b) There is no function $f : \mathbb{R} \to \mathbb{R}$, which is continuous exactly in the rational numbers.
